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Abstract

I study Schumpeterian growth under Variable Demand Elasticity preferences in a canonical Two-R&D-sector model with both vertical and horizontal innovation. Within this framework, I show how the departure from the traditional CES specification alters both the positive and normative characteristics of the Schumpeterian growth dynamics: (a) for a sufficiently high population growth rate relative to the innovation opportunity, there is a balanced growth path -"BGP"- of drastic innovation where the innovation size is determined by the population growth rate, that is growth is semi-endogenous. However, for a sufficiently low population growth rate, the model economy converges to the limit values of demand elasticity and to fully endogenous growth (b) Along the BGP with innovation size equal to the population growth rate, welfare is maximized with a higher ratio of product varieties per consumer and a lower per-variety output.

JEL Classification: O-30, O-40

Key-words: Schumpeterian Growth, Variable Demand Elasticity, Population Growth and Technological Progress.

1 Introduction

This work studies Schumpeterian growth under Variable Demand Elasticity ("VDE") preferences in a two-sector-R&D model and shows how departing from the traditional Constant Elasticity of Substitution (CES) and Constant Demand Elasticity (CDE) specification alters both positive and normative properties of Schumpeterian growth dynamics.[1] Since introduced by Dixit and Stiglitz (1977), the monopolistic competition with CDE has been a prominent framework in the IO literature and in the study of international trade and R&D-based growth.[2] Although highly tractable and convenient, the CDE specification lacks flexibility and yields some results that are at odds with empirical finding and common economic intuitions.[3] A recent research line aims to

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[1] For the sake of consistent terminology, hereafter I will refer to the CES specification as CDE and will compare it with the elaborated VDE specification studied here.

[2] Either CDE preferences over a variety of consumption goods, or CDE technology that uses variety of intermediate (or investment) goods.

[3] See Zhelobodko et al. (2012), page 2765.

explore the implications of more general - VDE - preference specifications in a static monopolistic-competition frameworks,[4] and a following sequence of studies implemented those VDE specifications in Romer's (1990) framework of R&D-based growth with horizontal innovation (variety expansion); see Bucci and Matveenko (2017), Boucekkine et al. (2017), and Etro (2018, 2019, 2023), Latzer at al. (2019) and Morita (2023). To the best of my knowledge this study is the first to incorporate VDE specification into a Schumpeterian growth model with vertical innovation.

The present analysis is carried within Young's (1998) two-sector R&D model economy with both vertical and horizontal innovation, as in Dinopoulos and Thompson (1998), Peretto (1998), Howitt (1999), and Segrestrom (2000).[5] The original analyses of these models were confined to the CDE specification, which is replaced here with the general additive separable preferences studied in Zhelobodko et al. (2012).[6] Moreover, unlike the previous analyses that focused on drastic innovation, The present analysis incorporates both drastic and non-drastic quality innovation.[7] Within this framework I first show that the size of drastic innovation, that is the rate of product-quality improvements, along the balanced growth path ("BGP") is determined by the population growth rate, which also dictates the product variety span expansion rate. Interestingly, along the BGP with drastic innovation individual per-variety consumption increases only from diversifying the consumption bundle over the ever expanding product variety span, while the utility derived from each variety remains stationary. Nonetheless, such a BGP is viable only for sufficiently high population growth rate, relative to the innovation opportunity that is measured by the innovation cost elasticity. For lower population growth rates quality innovation is non-drastic and independent of the population growth rate, and balanced growth can sustain only through convergence to the limit value of demand elasticity, that is convergence to the CDE limit-case that was studies by Young (1998) and in the other aforementioned two–sector–R&D models. There, innovation size along the BGP is determined by the demand elasticity parameter and innovation cost parameters, and population growth rate determines only the product-variety span expansion rate. Then, I show that for the BGP along which innovation size is equal to the population growth rate, welfare can be maximized by decreasing per-variety output and increasing per-consumer product varieties, relative to their values along the equilibrium BGP.

[4] See for example Zhelobodko et al. (2012) and Bertoletti and Etro (2015, 2016, 2021), Matsuyama and Ushchev (2022).

[5] The pioneering Schumpeterian growth models of vertical innovation by Grossman and Helpman (1991) and Aghion and Howitt (1992), had a fix number of product lines.

[6] In the working paper that first presented his two-sector-R&D model, Young (1995) considers the possibility of a variable elasticity of demand with respect to quality provision (see Section IV, p.18, there). However, he does that based on different micro foundations - building on Salop's circular market model in which the equilibrium prices (and markups) are independent of quality provision. In the journal article, Young (1998) also refers to the possibility of variable demand elasticity with respect to quality, but still abstracts the possibility that the markups may depend on innovation size and that demand elasticity may as well change with consumed quantity - see equations (11) and (25) and the following discussions, there. The current analysis provides a direct complete generalization of Young's (1998) model - from CDE to VDE specification.

[7] Drastic and non-drastic innovation are considered here as defined by Aghion and Hewitt (1992): with drastic innovation the quality improvement is large enough to allow the entrant to take over the product line while setting the monopolistic price according to demand elasticity. Under non-drastic innovation, prices are set through vertical (Bertrand) competition between the entrant and the incumbent of each product line.

The implications of population size and growth rate to technological progress have been central to the R&D-based growth literature from its very beginning. The first-generation models presented the "strong scale effect", that is a positive effect of population size on the R&D-based growth rate (Romer 1990, Grossman and Helpman 1991, Aghion and Howitt 1992). Jones (1995a,b), was the first to point at a lack of empirical support to the strong scale effect hypothesis, and proposed an alternative R&D-based growth model of horizontal innovation in which technological progress is positively affected by population growth rate (rather than population size). This "weak scale effect" that presents also in the aforementioned two-sector-R&D models, have regained researchers' attentions in light of the recent apparent slowdown in the pace of technological progress that is companioned with decreasing population growth rates in the US and other developed economies.[8][9]

The relation between innovation size and population growth rate has also immediate implications for the potential effectiveness of industrial policies, e.g., R&D subsidies and the design of patents, which were extensively studied in the growth literature:[10] If the exogenous population growth rate determines the rate of technological progress, growth is considered semi-endogenous and industrial-policy interventions are futile, as in Jones (1995b) model. If technological progress is independent of population growth rate and is affected by industrial policy, growth is considered fully endogenous, as in the two–sector–R&D models under the CDE specification. Cozzi (2017a,b) provides concise summary of the topic and proposes two hybrid models, each of which synthesizes R&D technology assumed by Jones (1995) and the one assumed in the two–sector–R&D models.[11][12] Under both specifications, there is a population growth rate threshhold above (below) which the economy converges to semi-endogenous (fully endogenous) growth. In an earlier related study, Li (2000) shows that the full growth endogeneity in the two–sector–R&D models relies on two knife edge assumptions regarding the innovation technology that eliminate knowledge spillovers between vertical and horizontal innovation. The results derived here show that the CDE specification serves as another knife-edge assumption that makes growth fully endogenous. However, in line with Cozzi's (2017a,b) results, a sufficiently high population growth rate is needed to sustain the VDE specification and the implied semi-endogenous growth viable.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes the consumer's and producer's optimization and the general equilibrium conditions. Section 4 characterized the decentralized and welfare maximizing growth dynamics, and Section 5 concludes this study.

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[8] See Jones (1999) for a compact comparative summary of the different model types.

[9] Jones (2022) provides an updated comprehensive review on the topic and Peters and Walsh (2022) present and calibrate a model of population growth and Schumpeterian growth.

[10] See Chu (2022) for a recent survey of the literature on patent policy in the R&D-based growth models.

[11] In Cozi (2017a), the total productivity growth is the weighted average of the two innovation technology outcomes, and in Cozi (2017b) the total productivity growth is a CES function of the two types of innovation technologies, which may be either substitutes or complements.

[12] Peretto (2015) characterizes the dynamics of economic development that include transition between the semi-endogenous and the fully endogenous regimes.

2 The Model

I employ Young's (1998) Schumpeterian growth model with one expositional modification: I replace the original production function of a composite final good that uses differentiated intermediate goods of improving quality, with a utility function over a variety of consumption goods of improving quality.[13] Then, I generalize the originally assumed CDE specification to the VDE direct utility function studies in Zhelobodko et al. (2012). All other specifications of Young's (1998) model economy remain, and thus all the results presented below coincide with Young's (1998) once demand elasticity is re-assumed to be constant. Time is discrete and in each period t the economy is populated with L_t homogenous infinitely lived agents, and population size expands at a constant rate, $n \equiv \frac{\triangle L_{t+1}}{L_t}$. Each worker supplies one unit of labor, so within each period population size equals aggregate labor supply.

2.1 Preferences

The lifetime utility of the representative consumer is given by

$$U = \sum_{t=0}^{\infty} \beta^t \ln(C_t) \tag{1}$$

where $\beta \in (0,1)$ is the time preference parameter. The intra-period consumption in (1), denoted C_t, is derived from M differentiated products, i.e. "varieties", subject to the direct additive utility function studied in Zhelobodko et al. (2012):

$$C_t = \int_0^{M_t} u\left(c_{i,t}\right) di \tag{2}$$

The sub-utility function $u(\bullet)$ is concave and thrice differentiable. The consumption stream derived from each product variety, c_i, is given by $c_i = q_i x_i$, where x_i and q_i are the utilized quantity and quality, respectively. It will be shown below that the utility function (2) implies the following variable demand elasticity, ε_{x_i,p_i}, and its absolute value $s(c_i)$, for each product variety[14][15]

$$s(c_i) \equiv |\varepsilon_{x_i,p_i}| = -\frac{u'(c_i)}{c_i u''(c_i)} \tag{3}$$

For equal consumption levels from all varieties, $s(c_i)$ defines also the elasticity of substitution across different varieties. Demand elasticity may be decreasing or increasing with consumption level, abbreviated DDE or IDE, respectively.[16] The value of demand elasticity is assumed to be

[13] These production function and utility function specifications are equivalent for the purposes of the current analysis and correspond the modeling choices of the canonical Schumpeterian growth models of Aghion and Hewitt (1992) and Grossman and Helpman (1991), respectviely.

[14] I will use "epsilon", ε, to denote all other elasticities considered below.

[15] For $u(c_i) = c_i^{\rho}$ equation (2) reduces back to the CDE case, with the demand elasticity equal to $\frac{1}{1-\rho}$.

[16] Zhelobodko et al. (2012) term the inverse of demand elasticity as the Relative Love for Variety, $RLV_i \equiv$

finite and greater than one, $s(c_i) \in (1, \infty)$, so product varieties are imperfect substitutes.

2.2 Technologies

Labor is the sole input for production and innovation, and the wage is normalized to one. One unit of labor produces one consumption good (regardless of its variety and quality). The two latter assumptions imply a unit marginal cost of production. Innovation is certain and takes one period to complete, subject to the following cost function[17]

$$f(q_{i,t+1}) = f\left(\frac{q_{i,t+1}}{\overline{q}_t}\right) \tag{4}$$

The innovation cost incured in period t, for enhencing the quality of product variety i, will yield an improved product to be commercialized in period $t + 1$.The innovation increases with the rate of quality improvement over the existing quality frontier, denoted $\overline{q}_t$, which is the highest product quality already attained in the economy.[18] To enhance tractability, let $\kappa_{i,t+1} \equiv \frac{q_{i,t+1}}{\overline{q}_t}$ denote the relative product quality over two consecutive periods. Hereafter, I will refer to κ as innovation size, which also defines the rate of quality improvement given by $\kappa - 1$, and to economies on notation I will denote $f(\kappa_{i,t+1})$ as $f_{i,t+1}$. Due to the assumed certain outcome of R&D investments, innovation takes exactly one period, and therefore the effective market lifetime of each quality improvement is one period as well, before being driven out of business by the developer of the next quality improvement - that is through the creative destruction process. This short firm lifetime greatly simplifies the profit maximization problem and thereby the equilibrium-dynamics analysis.

3 Equilibrium

3.1 Consumers' optimization

Lifetime utility (1) is maximized under the standard inter-temporal budget constraint

$$a_{t+1} = \frac{1 + r_{t+1}}{1 + n} a_t + w_t - e_t \tag{5}$$

where a denotes the consumer's assets that are held in form of patent ownership, r is the interest rate, and w is labor income that is normalized to one, and e the consumer's expenditure on consumption. The maximization of (1) under the composite consumption stream specification (2), the dynamic constraint (5), and assumed cross-variety symmetry, yields the following modified

$-\frac{c_i u''(c_i)}{u'(c_i)} = -\frac{1}{s_i(c_i)} > 0$, which corresponds the Prath-Arrow measure of relative risk aversion. If demand elasticity decreases (increases) the relative love for variety increases (decreases) with consumption level. Intuitively, an increasing (decreasing) demand elasticity across product varieties implies a higher (lower) elasticity of substitution across product lines, and thereby the consumer's gain from ("love for") additional variety increases.

[17] Young's (1998) analysis focuses on the exponential innovation cost, $f(q_{i,t+1}) = \exp\left(\phi\frac{q_{i,t+1}}{\overline{q}_t}\right)$, for which the innovation cost elasticity is linear: $\varepsilon_{f,\kappa} = \phi\frac{q_{i,t+1}}{\overline{q}_t}$, where $\phi > 0$ is a cost parameter.

[18] In Sorek (2024) I present a static-equilibrium analysis with a more general product cost function.

Euler condition over the consumer's inter-temporal spending path

$$\frac{e_{t+1}}{e_t}\frac{\varepsilon_{u,c_t}}{\varepsilon_{u,c_t+1}} = \frac{\beta\left(1+r_{t+1}\right)}{1+n} \tag{6}$$

where $\varepsilon_{u,c_t} \equiv \frac{c_t u'(c_t)}{u(c_t)}$, is the elasticity of the subutility function from each variety with respect to the consumption level $c_{i,t}$.[19] Under the assumed consumers' homogeneity, condition (6) can be written also in terms of aggregate consumers' spending, $E_t \equiv e_t L_t$:

$$\frac{E_{t+1}}{E_t}\frac{\varepsilon_{u,c_t}}{\varepsilon_{u,c_t+1}} = \beta\left(1+r_{t+1}\right) \tag{7}$$

Within each period, consumers maximize the instantaneous utility function (2) by allocating their spending over the different available product varieties, according to the following per-variety inverse demand function

$$p_{i,t} = \frac{q_{i,t}u'\left(q_{i,t}\cdot x_{i,t}\right)}{\lambda_t} \tag{8}$$

where λ_t is the Lagrange multiplier from the standard static consumer's optimization problem. The inverse demand function (8) implies the variable demand elasticity that was introduced already in (3):

$$\varepsilon_{x_i,\, p_i} = \frac{dx_i}{dp_i}\frac{p_i}{x_i} = \frac{u'\left(c_i\right)}{c_i u''\left(c_i\right)}$$

3.2 Profit maximization

Following Young (1998), the present analysis assumes complete lagging-breadth patent protection and no leading breadth protection nor minimal patentability requirement. Under these assumptions, the price set by the entrant must not exceed its innovation size, that is $p_i \leq \kappa_i$, so they can take over the market held by the incumbent that is the developer of the previous state-of-the-art product.[20] If the monopolistic profit-maximizing price satisifes the latter condition, the innovation is drastic and the quality-to-price ratio is sufficient to drive the previous product lines leader (the incumbent) out of business. Otherwise, the innovation is non-drastic, and the quality-to price-ratio implies a binding vertical competition between the entrant and the incumbent within each product line. With CDE preferences, for a sufficiently high (low) value of the exogenous demand elasticity, relative to the innovation cost parameters, innovation is drastic (non-drastic) and the profit maximizing price and innovation size are set independently (jointly). The following analysis shows how the profit-maximizing innovation-size and price are jointly determined under VDE preferences, along with demand elasticity. It starts with the case of non-drastic innovation and proceeds to drastic

[19] Under the CDE specification the subutility elasticity is constant and therefore is canceled out of the Euler condition.

[20] As to satisfy the vertical bertrand competition condition: $\frac{q_{t+1}}{p_{t+1}} \geqslant \frac{q_t}{MC}$, and for unit marginal cost: $\kappa_{t+1} \equiv \frac{q_{t+1}}{q_t} \geqslant p_{t+1}$.

innovation. As already noted, the certain innovation technology implies that the firm's profit-maximization problem span over two periods only: in the first period R&D investment in product quality enhancement is made, and in the following period the product is commercialized and yields a surplus.

3.2.1 Non-Drastic Innovation

With non-drastic innovation the product price is equal to the innovation size, $p_{i,t} = \kappa_{i,t}$. Therefore, given that the marginal cost of production is normalized to one, each innovating firm that plans to enter the market in period t maximizes the following present value profit, denoted Π, in period $t-1$:

$$\Pi_{i,t-1} = \frac{(\kappa_{i,t} - 1)\, x_{i,t} L_t}{1 + r_t} - f_{i,t} \tag{9}$$

As demand elasticity is directly determined by the individual quantity and quality utilization levels, I present the profit maximization in terms of x and κ, whereas $y_{i,t} \equiv x_{i,t} \cdot L_t$ is the firm's output. The first order condition for maximizing (9) with respect to κ is[21]

$$\frac{\left[1 + \frac{(\kappa^*_{i,t} - 1)}{x_{i,t}} \frac{dx_{i,t}}{d\kappa_{i,t}}\right] x_{i,t} L_t}{1 + r_t} = f'_{i,t} \tag{10}$$

Condition (10) equalizes the present value of the marginal profit from innovation size with the marginal cost of innovation size. Combining (10) with the free-entry condition, that is imposing zero profit in (9), yields the following equilibrium innovation size[22]

$$\frac{1}{\kappa_{i,t} - 1} + \frac{dx_{i,t}}{d\kappa_{i,t}} \cdot \frac{1}{x_{i,t}} = \frac{f'_{i,t}}{f_{i,t}} \tag{11}$$

After setting $p_{i,t} = \kappa_{i,t}$ in (8), I can rewrite the inverse demand function as $q_{i,t-1}\kappa_{i,t}u'\left(q_{i,t-1}\kappa_{i,t} \cdot x_{i,t}\right) - \lambda_t \kappa_{i,t} = 0$, and from this equation I obtain the derivative of the individual demanded quantity with respect to product quality:

$$\frac{dx_{i,t}}{d\kappa_{i,t}} = -\frac{q_{i,t-1}u'\left(q_{i,t} \cdot x_{i,t}\right) + q_{i,t}q_{t-1}x_{i,t}u''\left(q_{i,t} \cdot x_{i,t}\right) - \lambda_t}{q_{i,t}q_{i,t}u''\left(q_{i,t} \cdot x_{i,t}\right)} \tag{12}$$

Then, substituting $\lambda_t = \frac{q_{i,t}u'(q_{i,t} \cdot x_{i,t})}{\kappa_t}$ from (8) into (12) yields[23]

$$\frac{dx_{i,t}}{d\kappa_{i,t}} \cdot \frac{1}{x_{i,t}} = -\frac{1}{\kappa_{i,t}} \tag{12a}$$

[21] The asterisk superscript denotes the values that maximize individual objective functions and the "e" superscript denotes equilibrium values.

[22] Condition (11) can be written also in elasticity terms: $\eta_{(p-1),\kappa} + \eta_{x,\kappa} = \eta_{f,\kappa}$.

[23] This results can be inferred directly by plugging $p_{i,t} = \kappa_{i,t}$ into the inverse demand equation, $\lambda_t = \frac{q_{i,t}u'\left(q_{i,t} \cdot x_{i,t}\right)}{\kappa_{i,t}}$, and then substitute $q_{i,t} = q_{i,t-1}\kappa_{i,t}$ into the latter equation to obtain $\lambda_t = q_{i,t-1}u'\left(q_{i,t-1}\kappa_{i,t} \cdot x_{i,t}\right)$, which implies a unit elasticity of demanded quantity with respect to innovation size.

Equation (12a) implies a negative unit elasticity of demanded quantity with respect to the size of non-drastic innovation, due to the combined effect of omproved product quality and a higher price increase that is associated with innovation size. Plugging (12a) back into (11) reveals that the profit-maximizing size of non-drastic innovation is independent of demand elasticity, and is time invariant:

$$\forall t, i : \frac{1}{\kappa^e - 1} = \varepsilon_{f,\kappa^e} \tag{13}$$

Equation (13) implies that in the equilibrium with non-drastic innovation the surplus elasticity with respect to innovation size is equal to the innovation cost elasticity.

3.2.2 Drastic innovation

For drastic innovation the profit maximizing price is smaller than innovation size. Given the inverse demand function (8), each innovating firm maximizes the following present-value profit

$$\Pi_{i,t-1} = \frac{\left(\frac{q_{i,t}u'(q_{i,t}\cdot x_{i,t})}{\lambda_t} - 1\right) x_{i,t}L_t}{1 + r_t} - f_{i,t} \tag{14}$$

The first order condition for maximizing (14) with respect to x_i reads[24]

$$q_{i,t}u'\left(q_{i,t} \cdot x_{i,t}\right) + x_{i,t}q_{i,t}^2 u''\left(q_{i,t} \cdot x_{i,t}\right) = \lambda_t \tag{15}$$

Combining (15) with the price equation (8) yields the following profit-maximizing price

$$p_{i,t}^* = \frac{1}{1 + \frac{xq_{i,t}u''(q_{i,t}\cdot x_{i,t})}{u'(q_{i,t}\cdot x_{i,t})}} \tag{16}$$

which can be also written as the familiar monopolistic-pricing rule

$$p_{i,t}^* = \frac{s(c_{i,t})}{s(c_{i,t}) - 1} \equiv \mu_{i,t} \tag{16a}$$

where $\mu > 1$ is markup meaure that decreases with demand elasticity.[25] Differentiating the profit (14) with respect to product quality yields the following first order condition:

$$\frac{\frac{u'(q_{i,t}\cdot x_{i,t})}{\lambda_t}\left(1 + \frac{q_{i,t}x_{i,t}u''(q_{i,t}\cdot x_{i,t})}{u'(q_{i,t}\cdot x_{i,t})}\right) x_{i,t}L_t}{1 + r_t} = f'_{i,t} \tag{17}$$

[24] Condition (15), which defines the surplus (and profit) maximizing quantity, can be also written as: $\frac{q_{i,t}u'(c_{i,t})(1-RLV_{i,t})}{\lambda_t} = 1$. The left side in this equation is the marginal revenue associated with the sale of each product unit and the right side is the marginal cost. Under the assumption $RLV_i < 1$, the marginal revenue is positive for any output level. For an increasing RLV the marginal revenue is guaranteed to decrease with output level, and assuming $\frac{c_{i,t}u'''(c_{i,t})}{u''(c_{i,t})} > -2$ guarantees that the marginal revenue is decreasing with output level also for a decreasing RLV.

[25] Without the unit-marginal cost normalization that applies here, the pricing rule (16a) reads $p = \mu \cdot mc$, where mc is the marginal production cost.

By (8), the term $\frac{u'(q_{i,t} \cdot x_{i,t})}{\lambda_t}$ in (17) can be also written as $\frac{p_{i,t}}{q_{t,i}}$, and substituting in (17) the profit maximizing price (16a) yields

$$\frac{x_{i,t} L_t}{1 + r_t} = \kappa_{i,t} f'(\kappa_{i,t}) \tag{17a}$$

Plugging the price (16) back into (14), and combining the zero-profit condition with the optimality condition (17a) yields the following free-entry equilibrium condition that applies to each product variety

$$s(c_t^e) - 1 = \varepsilon_{f_t, \kappa_t}^e \tag{18}$$

which can be also written in terms of the equilibrium price[26]

$$\mu\left(c_t^e\right) = \frac{1}{\varepsilon_{f_t, \kappa_t}^e} + 1 \tag{18a}$$

Condition (17a) implies $c_t \equiv q_t x_t = \frac{\overline{q}_{t-1} \kappa_t^2 f'(\kappa_t)(1+r_t)}{L_t}$. Therefore, condition (18) can be written in terms of the innovation size only

$$s\left(\frac{\overline{q}_{t-1} \kappa_t^2 f'(\kappa_t)\left(1 + r_t\right)}{L_t}\right) - 1 = \varepsilon_{f_t, \kappa_t}^e \tag{18b}$$

The equilibrium size of drastic innovation is determined by demand elasticity and the innovation-cost elasticity.[27] For DDE (IDE), the left side in (18a) decreases (increases) with innovation size. I assume that an equilibrium exists for DDE. For IDE, the existence of equilibrium it is necessary that innovation cost elasticity increases with innovation size.

3.3 Product variety span

Within each and every period the aggregate resources-uses constraint (that is also the labor market clearing condition) requires

$$L_t = \frac{E_t}{p_t} + M_{t+1} f(\kappa_{t+1}) \tag{19}$$

Equation (19) implies that aggregate labor supply is fully employed in production and R&D activity, which are the first and second addend on the right side of the equation, respectively. Applying the zero-profit condition and the Euler condition (7) to equation (19), yields

[26] In Sorek (2024) I show that for general production cost, $tc(y)$, the equilibrium condition is $\mu = \frac{1}{\varepsilon_{f,\kappa}} + \frac{1}{\varepsilon_{tc,y}}$. The markup measure μ is also the inverse of the elasticity of the revenue, R, with respect to consumption, that is $\mu \equiv \frac{s}{s-1} = \frac{1}{\varepsilon_{R,c}}$. The assumed constant marginal cost and no fix production cost, imply $\varepsilon_{tc,y} = 1$, and the resulting equilibrium condition (18).

[27] Under the CDE specification, $u(c_i) = c_i^{\rho}$, equation (18) boils down to $\frac{1}{1-\rho} - 1 = \varepsilon_{f,\kappa}^e$, where $\frac{1}{1-\rho}$ is the constant demand elasticity, as in Young (1998).

$$L_t = \frac{E_t}{p_t} + (1 - \frac{1}{p_t})\beta E_t \frac{\varepsilon_{u,c_t+1}}{\varepsilon_{u,c_t}} \tag{20}$$

Then, solving (19a) for E_t, and plugging it back into (19), along with the relevant price expressions, yields the following equilibrium variety span for non-drastic and drastic innovation, respectively:

$$M^e_{t+1} = \frac{L_t}{f^e \left[\frac{1}{(\kappa^e - 1)\beta \frac{\varepsilon_{u,c^e_{t+1}}}{\varepsilon_{u,c^e_t}}} + 1 \right]} \tag{20a}$$

$$M^e_{t+1} = \frac{L_t}{f^e_{t+1} \left[\frac{1}{\mu^e_t (1 - \frac{1}{\mu^e_{t+1}})\beta \frac{\varepsilon_{u,c^e_{t+1}}}{\varepsilon_{u,c^e_t}}} + 1 \right]} \tag{20b}$$

4 Growth

This section combines all the results derived thus far to characterize the equilibrium Balanced Growth Path, BGP, along which individual (per-capita) consumption increases at a steady rate, and compare it with the welfare maximizing BGP. The section concludes with a concrete example of VDE preferences, that is used to illustrate the main results.

4.1 Endogenous and semi-endogenous growth

The equilibrium condition (18) implies that a stationary innovation size requires a stationary individual per-variety consumption, which, by equation (16a), implies also a stationary products price. However, a stationary innovation size, interest rate, and individual per-variety consumption , $c_t = \frac{q_t}{L_t}\kappa_t f'(\kappa_t)(1 + r_t)$, implies a stationary ratio of product quality to population, $\frac{q_t}{L_t}$. Therefore, the steady quality growth rate is equal to the population growth rate $\kappa = 1 + n$.[28] This, in turn, implies - by equation (20) - that the product variety span also expands at the same rate: $1 + m = 1 + n = \kappa$. Having $m = n$, and stationary price and innovation size in equation (19), requires that along the BGP aggregate consumer spending also increases at the rate n, by the Euler condition (7) implies that the stationary interest rate is $1 + r = \frac{1+n}{\beta}$. Consequently, along the BGP the individual per-variety consumption is given $c_i^{BGP} = \frac{q_t}{L_t}\frac{(1+n)f'(1+n)}{\beta}$, and equilibrium condition (18) reads

$$s(\frac{q_t}{L_t}\frac{(1+n)f'(1+n)}{\beta}) - 1 = \frac{(1+n)f'(1+n)}{f(1+n)} \tag{21}$$

[28] For $1 + n > \kappa$ $(1 + n < \kappa)$ per-variety consumption is declining (increasing) over time. With CDE preferences, the innovation size and the price are necessarily time invariant and determined independently by the given demand elasticity, and the product variety span expands at the same rate as population growth. Consequently, the growth of variety consumption is $1 + g_{c_i} = \frac{\kappa^*}{1+m^*}$ and total consumption grows at the rate κ^*.

The individual per-variety consumption level along the BGP can satisfy condition (21) only for a unique quality-to-population- size ratio, $\frac{q_t}{L_t}$. Therefore, unless the initial condition $\frac{q_0}{L_0}$ happens to coincide with this value, balanced growth with VDE does not sustain instantly, but through convergence dynamics, unlike in the CDE case. Applying equation (21) to equation (20b) yields the product variety span per consumer along the BGP:

$$\frac{M_t^{BGP}}{L_t} = \frac{1}{(1+n)f(1+n)\left(\frac{\varepsilon_{f,\kappa=1+n}}{\beta}+1\right)} \tag{22}$$

However, a BGP with drastic innovation is viable only if the innovation size dictated by the population growth rate is larger than the corresponding monopolistic price, which defined by the consumption level c_i^{BGP}:

$$\kappa^{BGP} = 1+n \geq \underbrace{\frac{s(c^{BGP})}{s(c^{BGP})-1}}_{\mu(c^{BGP})} \tag{23}$$

Applying (18a) to substitute the right side of (23) for $\frac{1}{\varepsilon_{f,\kappa=1+n}}+1$, yields the following condition for the viability of the BGP with drastic innovation:

$$n \geqslant \frac{1}{\varepsilon_{f,\kappa=1+n}} \tag{23a}$$

For constant or increasing innovation-cost elasticity, there exists a unique population growth rate below (above) which condition (23a) does not hold (holds). The population growth rate that satisfies condition (23a) with equality coincides with the quality improvement rate that satisfies condition (13) for non-drastic innovation. This is the non-drastic innovation size that sustains under any lower population growth rate for which condition (23a) does not hold. In this case, balanced growth can be only attained at the limit value of demand elasticity, as the economy converges the CDE case, for which innovation size is independent of population growth rate and product variety span expansion rate is equal to the population growth rate, $m = n$.

Proposition 1 *For sufficiently high (low) population growth rate economic growth is semi (fully) endogenous: innovation size along the balanced growth path is drastic (non-drastic) and determined by population growth rate (by the innovation technology).*

4.2 Welfare

The socially-optimal rate of quality improvements and product variety span is defined by the allocation of labor over R&D activity and production that maximizes the lifetime utility (1), subject to the aggregate resources-uses constraint (19). Substituting (19) into (1) yields the welfare maximization objective function:[29]

[29] It is assumed that the transversality condition, $\underset{t\longrightarrow\infty}{Lim}\,\beta^t u(c_t) = 0$, holds.

$$U = \sum_{t=0}^{\infty} \beta^t \ln[M_t u(q_t x_t)] = \sum_{t=0}^{\infty} \beta^t \ln M_t u \left(q_{t-1}\kappa_t \left[\underbrace{\frac{L_t - M_{t+1} f(\kappa_{t+1})}{L_t M_t}}_{x_t} \right] \right) = \tag{24}$$

$$= \sum_{t=0}^{\infty} \beta^t \ln M_t + \sum_{t=0}^{\infty} \beta^t \ln u \left(q_{t-1}\kappa_t \left(\underbrace{\frac{L_t - M_{t+1} f(\kappa_{t+1})}{L_t M_t}}_{x_t} \right) \right)$$

The first order conditions for maximizing (24) with respect to κ_t and M_t, satisfy[30]

$$\frac{\varepsilon_{u,c_t^{**}} + \sum_{j=0}^{\infty} \beta^{j+1} \varepsilon_{u,c_{t+j}^{**}}}{1 - \varepsilon_{u,c_t^{**}}} = \varepsilon_{f,\kappa_t^{**}} \tag{25a}$$

$$M_{t+1}^{**} = \frac{L_t}{f(\kappa_t^{**}) \left[\frac{\varepsilon_{u,c_{t-1}^{**}}}{\left(1-\varepsilon_{u,c_t^{**}}\right)\beta} + 1 \right]} \tag{25b}$$

and for a steady individual per-variety consumption level the above efficiency conditions read

$$\frac{\varepsilon_{u,c^{**}}}{(1 - \varepsilon_{u,c^{**}})(1 - \beta)} = \varepsilon_{f,\kappa^{**}} \tag{26a}$$

$$M_{t+1}^{**} = \frac{L_t}{f(\kappa^{**}) \left[\frac{\varepsilon_{u,c^{**}}}{(1-\varepsilon_{u,c^{**}})\beta} + 1 \right]} \tag{26b}$$

The welfare maximizing market outcomes defined in (25a)-(25b) and (26a)-(26b) depend on the elasticity of the subutility function whereas the corresponding equilibrium outcomes in (18) and (20) depend on demand elasticity. This general divergence between the welfare maximizing conditions and the free-entry equilibrium conditions was already established by Dixit and Stiglitz (1977) in the static monopolistic competition framework. Here, however, additional - dynamic - inefficiency rises due to the divergence between the infinite optimization horizon faced by the social planner and the limited profit-maximization horizon of the firms: the efficiency conditions (25a)-(26a) incorporate all future values of the utility elasticity counting for the vertical-dynamic knowledge spillover associated with quality improvements, whereas the equilibrium condition (18) include only the demand elasticity value during the single period lifetime of each product (and firm). For $\beta = 0$, conditions (25a)-(26a) coincides with the ones derived in Sorek (2024) for optimal product-quality provision in a static analysis of the present framework.[31] In the CDE case, $\varepsilon_{u,c}$ is also constant, and $\frac{\varepsilon_{u,c}}{1-\varepsilon_{u,c}} = s - 1$. Consequently, the welfare maximizing innovation size, defined

[30] The socially-optimal values are denoted with double asterisk super script.

[31] See equation (16) there, after setting the production cost elasticity to one, as in the present analysis.

by $\frac{s-1}{1-\beta} = \varepsilon^{**}_{f,\kappa}$, is larger than the equilibrium drastic innovation size, defined by $s - 1 = \varepsilon^{e}_{f,\kappa}$. With VDE, for a given per-variety consumption level, the expression $\frac{\varepsilon_{u,c}}{1-\varepsilon_{u,c}}$ in the left side of (26a) may be higher or lower than $s(c) - 1$ from the left side in (18), depending on whether $\varepsilon_{u,c}$ increases or decreases with consumption level, respectively. As a result, with $\beta = 0$ per-variety consumption level along the welfare maximizing BGP is lower than along the equilibrium BGP, that is $c^{**} < c^{BGP}$, but with sufficiently high β the reverse inequality holds.

Furthermore, with VDE, a welfare maximizing BGP also requires the rates of product quality improvements and product variety expansion to equal the population growth rate, just as along the equilibrium BGP. To see that, substitute (26a) into (26b) to obtain the following expression for the product varieties per-consumer ratio

$$\frac{M^{**}_{t+1}}{L_{t+1}} = \frac{1}{f(\kappa^{**})(1+n)\left[\frac{1-\beta}{\beta}\cdot\varepsilon_{f,\kappa^{**}} + 1\right]} \tag{25}$$

Applying (27) to the explicit expression of c_i in (24) yields the following individual per-variety consumption level along an efficient BGP:

$$c^{**} = \frac{q_t}{L_t}\frac{(1+m)(1-\beta)f'(\kappa^{**})}{\beta} \tag{26}$$

Equation (28) shows that, as in the equilibrium growth dynamics, along an efficient steady growth path the quality improvement rate must be equal to the population growth rate, $\kappa = 1+n$. This, in turn, implies that the expansion rate of product-variety span must also expand at the same rate: $m = n$. Under this requirement conditions (27) and (28) read

$$\frac{M^{**}_{t}}{L_{t}} = \frac{1}{(1+n)f(1+n)\left(\frac{1-\beta}{\beta}\cdot\varepsilon_{f,\kappa=1+n} + 1\right)} \tag{27a}$$

$$c^{**} = \frac{q_t}{L_t}\frac{(1+n)(1-\beta)f'(1+n)}{\beta} \tag{28a}$$

Along the welfare maximizing BGP, the individual per-variety consumption level in (28a) should equalize the left side in (26a) to the innovation cost elasticity evaluated at $\kappa = 1+n$. If there is a consumption level c^{**} that satisfies this condition, there is also a product quality to population size ratio, $\frac{q_t}{L_t}$, that support this consumption level. Comparing the per-variety consumption level in (28a) with its counterpart along the equilibrium BGP, $c_i^{BGP} = \frac{q_t}{L_t}\frac{(1+n)f'(1+n)}{\beta}$, reveals that per-variety output along the efficient BGP is smaller than its counterpart along the equilibrium BGP: $y^{**} \equiv \frac{L_t c^{**}}{q_t} = \frac{(1+n)(1-\beta)f'(1+n)}{\beta} < y^{BGP} = \frac{(1+n)f'(1+n)}{\beta}$. Moreover, comparing (27a) with (22), reveals that along the welfare maximizing BGP the per consumer product-varieties ratio is greater than its counterpart along the equilibrium BGP.

Proposition 2 *There exists a welfare maximizing BGP along which innovation size is equal to the population growth rate, the number of product varieties per consumer is larger than along the* *equilibrium BGP with drastic innovation, and per-variety output is small than along the equilibrium BGP.*

Innovation size along the efficient BGP is determined by the population growth rate, even for values below the threshold level for which the decentralized economy switches to the single non-drastic innovation size. Therefore, innovation size along the equilibrium BGP with non-drastic innovation is greater than the innovation size along the welfare maximizing BGP.[32]

4.3 Example

To illustrate the main results derived thus far, consider the following example of a VDE subutility function

$$u\left(c_{i,t}\right)=\frac{\left(A+Bc_{i,t}\right)^{\frac{B-1}{B}}}{B-1} \tag{27}$$

For $B\neq 1$ and $A=0$, the subutility function (29) boils down to the familiar CDE specification. With $B\neq 1$ and $A\neq 0$, we have[33]

$$\begin{aligned}\varepsilon_{u,c} &\equiv \frac{c_{i,t}u'\left(c_{i,t}\right)}{u\left(c_{i,t}\right)}=\frac{B-1}{\frac{A}{c_{i,t}}+B} \\ s_i(c_i) &\equiv -\frac{u'\left(c_{i,t}\right)}{c_{i,t}u''\left(c_{i,t}\right)}=\frac{A}{c_{i,t}}+B\end{aligned} \tag{29a}$$

Notice that as consumption level approaches infinity all the measures presented above converge to their CDE specification value defined by $B>1$. Consider the case with $A>0$, for which demand (subutility) elasticity decreases (increases) with consumption level, ranging from infinity (zero) to B ($\frac{B-1}{B}$). For the innovation cost function, consider the exponential form employed also in Young (1998):

$$f(\kappa)=\exp(\phi\kappa) \tag{28}$$

with $\phi>0$. Under the assumed specifications of the subutility and innovation cost functions, the equilibrium condition (18) for BGP with drastic innovation reads

$$\kappa^{BGP}=\frac{A}{c^{BG}}+B-1=\phi\left(1+n\right)$$

Implying the BGP per-verity consumption level

[32] In Sorek (2021) I show that for sufficiently low demand elasticity values of CDE preferences, non-drastic innovation exceed the welfare maximizing innovation size.

[33] For $B=-1$ the utility function (22) takes the quadratic form $u\left(c_{i,t}\right)=-\left(A-c_{i,t}\right)^2$, and for $B=1$ it converges to the logarithm form $u\left(c_{i,t}\right)=\ln\left(A+c_{i,t}\right)$.

$$c^{BGP} = \frac{A}{\phi\,(1+n) - (B-1)}$$

and the corresponding BGP equilibrium price

$$p^{BGP} = 1 + \frac{1}{\phi\,(1+n)}$$

Having a positive per-variety consumption level along the BGP, $c^{BGP} > 0$, requires $\phi > \frac{B-1}{1+n}$. Having drastic innovation along the BGP requires $p^{BG} < \kappa^{BGP} = 1+n$, that is $\frac{1}{\phi} < n\,(1+n)$. Given the subutility parameter B, for sufficiently high population growth rates relative to the innovation cost parameter, there exists a balanced growth path with drastic innovation along which the product quality per population size and product variety span per population size satisfy

$$\frac{q_t^{BGP}}{L_t} = \frac{A\beta}{[\phi\,(1+n) - (B-1)]\,\phi(1+n)\exp[\phi(1+n)]}$$

$$\frac{M_t^{BGP}}{L_t} = \frac{1}{(1+n)\exp[\phi(1+n)]\left(\frac{\phi(1+n)}{\beta}+1\right)}$$

With $\phi > B-1$, the per-variety consumption level along the BGP defined above is positive for any non-negative population growth rate. However, for $n \in \left[0, \sqrt{\frac{1}{4}+\frac{1}{\phi}} - \frac{1}{2}\right]$ innovation becomes non-drastic, with $\kappa = \frac{1+\sqrt{1+\frac{4}{\phi}}}{2} > 1$, and balanced growth can be attained only through convergence to the CDE limit-case, with $s = B$, as per variety consumption level approaches infinity.

Under the assumed preferences and innovation cost specifications, the efficiency condition (26a) along the BGP reads

$$\frac{B-1}{\left(\frac{A}{c^{**}}+1\right)(1-\beta)} = \phi\,(1+n)$$

and the implied efficient individual per-variety consumption along the BGP is

$$c^{**} = \frac{A}{\frac{B-1}{\phi(1+n)(1-\beta)} - 1}$$

Having per-variety consumption level positive along both the equilibrium and welfare maximizing BGP requires: $\frac{B-1}{(1+n)} < \phi < \frac{B-1}{(1+n)(1-\beta)}$. Comparing the above expression for c^{BGP} with the one for c^{**} reveals that for sufficiently low (high) value of β the welfare maximizing per-variety consumption level is higher (lower) than its equilibrium counterpart. The welfare maximizing quality-to-population ratio and per consumer product varieties along the BGP are given by

$$\frac{q_t^{**}}{L_t} = \frac{A\beta}{\left[\frac{B-1}{\phi(1+n)(1-\beta)} - 1\right] \phi(1+n)^2 \exp[\phi\,(1+n)]\,(1-\beta)}$$

$$\frac{M_t^{**}}{L_t} = \frac{1}{(1+n)\exp[\phi(1+n)]\left[\frac{\phi(1-\beta)(1+n)}{\beta} + 1\right]}$$

5 Conclusion

This work studies Schumpeterian growth in a two-sector-R&D economy with Variable Demand Elasticity preferences. It shows how the departure from the traditional Constant Demand Elasticity specification alters both the positive and normative characteristics of the Schumpeterian growth dynamics: the effect of population growth on innovation size and the potential effectiveness of industrial policy, and the possible deviation of the equilibrium BGP from the welfare maximizing one. Moreover, it shows that the BGP with VDE is viable only for drastic innovation that is supported by a sufficiently high population growth rate, relative to the innovation opportunity. Once the population growth rate falls below a (positive) threshold level innovation becomes non-drastic and the economy converges to its CDE limit values.

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