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Abstract

We show how the two alternative saving motives, life-cycle consumption smoothing and parental bequests, determine the relation between population growth and R&D-based economic growth, i.e. the sign of the weak scale effect. We take a textbook R&D-based growth model of infinitely living agents with no life-cycle saving motive and re-analyze it in the Overlapping Generations (OLG) framework, which incorporates both life-cycle and bequest saving motives. We decompose the effect of each saving motive on the sign of the weak scale effect, and show that in the presence of both saving motives it is ambiguous in general, and may also be non-monotonic. Hence, this study contributes to the recent line of research aimed to align modern growth theory with the empirical evidence on the relation between population growth and economic prosperity.

JEL Classification: O-31, O-40

Key-words: R&D-based Growth, Weak Scale-Effect, Bequests, OLG

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†We thank Pietro Peretto for helpful comments. We started studying the current topic in an earlier circulated working paper, titled "Finite Lifetimes Population and Growth". That paper presented different modeling approach, which we abandoned after detecting some analytical flaw. An earlier version of the present work, titled "Life-Cycle Saving, Bequests, and the Role of Population in R&D-based Growth", was also circulated.

1 Introduction

The second and third generations of R&D-based growth models were criticized for presenting a positive relation between population growth and economic prosperity, i.e. "weak scale effect, which does not fit the empirical findings of an ambiguous, and possibly non-monotonic, relation between these variables[1]. A recent line of research has proposed several modifications, aimed to align the theory with the empirical evidence. A common element in these modified models is the introduction of human capital as a productive input in the R&D sector[2]. Following the seminal works of Romer (1990), Grossman and Helpman (1991a,b) and Aghion and Howitt (1992), this literature has focused, almost exclusively, on the analysis of infinitely living homogenous agents.

Nevertheless, recent exceptional works, by Strulik et al.(2013) and Prettner (2014), have studied the relation between population growth and innovation based prosperity in the Overlapping Generations (OLG) framework of finitely living agents. Prettner (2014) shows that the relation between fertility rate and economic growth may depend on the provision of public education: teachers' productivity in the sector and per-student spending[3]. Strulik et al.(2013) developed a unified growth model that incorporates endogenous and human capital accumulation fertility and transition from neoclassical technology to R&D-based growth[4].

The two canonical demographic structures of the macroeconomic workhorse models, imply different incentives for saving. In the present work, we emphasize that these different saving motives have, by themselves, immediate implications to the presence of weak scale effect in R&D growth models. To this end, we take a standard textbook R&D-growth model, which was written for infinitely living agents with no human capital accumulation, and place it in the OLG framework to derive comparable results[5].

The infinitely living agents are typically assumed to share their assets (patent ownership in the current context and physical capital in the neoclassical models) with their offspring. They fully internalize this into their saving decisions as they maximize the per-capita or aggregate lifetime utility of their dynasty members. Therefore, in this framework savings involve bequests, but they lack a life-cycle saving motive as workers' labor supply does not change with age[6].

[1] Seminal models of the second and third generations models are Jones 1995, Kortum 1997, and Segerstrom 1998, and Peretto 1998, Young 1998 and Howitt 1999, respectively. Jones 1999 provides a compact comparative summary of this literature. See Strulik et al. (2013) and Boikos et al.(2013) for recent summaries of the empirical literature.

[2] See for example Dalgaard and Kreiner (2001), Strulik (2005), Bucci (2008, 2015), and Diwakar and Sorek (2016a,b). Boucekkine et al.(2013) provide extensive analysis of the effect of population size and growth rate on per-capita output level and growth rate, in an Uzawa–Lucas type model of infinitely living agents with both human and physical capital accumulation, but R&D-driven innovation.

[3] Diwakar and Sorek (2016b) present a similar mechanism for private houesholds' human capital formation.

[4] Two other recent works, by Prettner (2013) and Prettner and Timborn (2016), study the overall effect of combined decline in mortality and fertility on R&D based growth, within the perpetual-youth model of homogenous agents. They show that the positive effect of longevity gains on growth may over come the negative effect of decreasing fertility on growth.

[5] Earlier literature already showed that the different demographic structures have immediate implications on other key issues in neoclassical growth models, such as tax policy, convergence patterns, and the feasibility of growth itself. Dalgaard and Jensen (2009, p.1639) summarize this literature. Sorek (2011), and Diwakar and Sorek (2016c) highlight the implications of the OLG demographic structure to patent policy.

[6] The infinitely living agents can be thought equivalently, and more realistically, as finitely living ones with strong

By contrast, in the standard OLG framework saving is aimed to smooth consumption over a finite lifetime, which spans from working years to retirement period, and there are no intergenerational bequests. Hence, in this framework saving is motivated purely by life-cycle considerations. Clearly, the exclusive presentation of each saving motive in its corresponding demographic structure is unrealistically extreme[7].

Our analysis decomposes the implications of the two saving motives to the presence of weak scale-effect in a second-generation variety expansion model, and then considers their combines effect. We find that in the absence of bequest saving-motive, the effect of population growth on economic growth depends solely on the inter-temporal elasticity of substitution (*IES*): it is positive (negative or zero) if the *IES* is greater than (lower than or equal to) one[8]. This result is presented in Proposition 1.

In comparison, the corresponding model with infinitely living agents (with no altruism) yields no-relation between population growth and economic growth regardless of the *IES* value. In both models, population growth increases future demand for patented machines, thereby increasing the equilibrium interest rate . However, for the infinitely living agents, population growth works also as a demographic discounting factor which discourages saving[9], and thus the two effects cancel out. In the OLG economy, population growth does not generate direct negative effect on saving and, due to the life-cycle structure of the OLG framework, the effect of the increased interest rate on saving depends on the *IES*.

We then introduce bequest saving motive that is driven by the joy-of-giving, which, for convenience considerations, is common in the analyses of OLG models. In Proposition 2 we show that when the life-cycle saving channel is shut down, the sign of the weak scale-effect depends on the specification of parental utility from bequest-giving. Namely, it depends on the interaction between the level of bequest per-child and the number of kids in parents' utility: Parents may care only about their per-child bequest giving, in line with Millian specification. However, their utility from the level of per-child bequest may also increase or decrease with the number of children, consistently with Benthamite specification and the formulation emphasized by Barro and Becker (1989) and Becker et al.(1990), respectively. The latter specification forms a trade-off between the number of kids and per-child bequest level in parent's total utility from giving, which resembles a quality-quantity trade off in the overall joy-of-giving.

We find that when the Millian, Benthamite, and Beckerian types of the bequest-saving motive yield positive, negative, or no weak scale effect in our model economy, respectively. This is in line with the results derived the reference textbook model, as well as in by Dalgaard and Kreiner (2001), Strulik (2005), Boucekkine et al.(2013), Bucci (2013) and Diwakar and Sorek (2016b), for the infinite horizon framework with human capital accumulation.

altruism toward their offspring.

[7]the empirical literature has not yet reached an agreement regarding their relative importance in driving saving behavior; See De Nardi et al. (2015) for a recent survey.

[8]The empirical literature suggests that the IES is lower than one; See Hall (1988), Beaudry and Wincoop (1996), Ogaki and Reinhart (1998), Engelhardt and Kumar (2009).

[9]Following the standard Euler condition $\frac{\dot{c}}{c} = \frac{1}{\theta}\left(r - \rho - n\right)$. See extended explanation in Section 5.

There, however, the mechanism at work involves a tension between a positive effect of population growth on saving in the presence of dynastic altruism and its negative (diluting) effect on human capital accumulation[10].

Finally, in Proposition 3, we demonstrate that when both saving motives are active in determining the relation between population growth and economic growth, the actual sign of weak effect depends also on the relative strength of parents utility from bequest vs. utility from their own consumption during retirement.

Our study is also related to the work by Dalgaard and Jensen (2009), hereafter "*DJ*", on the effect of alternative saving motives on the presence of *strong scale effect* - that is the effect of population *size* on economic growth. They showed that population size has positive effect on growth when the bequest motive is dominant but it may turn negative when the life-cycle motive dominates, and may be non-monotonic as well. Their work adds bequest saving-motive to an otherwise standard OLG model with capital externalities, that is an AK model, and derives comparative statics with respect to a single bequest-motive parameter.

However, it should be emphasized that our research question differs from *DJ*'s, and that we employ alternative, more suitable, setup. Namely, we study the effect of alternative saving motives on the presence of *weak scale effect*, that is the relation between *population growth* and economic growth. To derive comparable results with the existing literature, written for infinitely living agents, we incorporate a full-fledged textbook model of R&D-based growth within the OLG framework. Moreover, as we study the effect of fertility rate on growth, in relation to the bequest saving-motive, we are required to address the interaction between the level of bequest per-child and the number of kids in parents' utility. This specification, which is not relevant for *DJ*'s analysis of *strong scale effects*, proves to be crucial to the presence of *weak scale effects*.

Due to these differences the topic under study (i.e., *strong* vs. *weak* scale effects), and modeling approach, our results are not fully comparable with those of *DJ*. However, in Section 5 we will show that *DJ*'s conditions for the presence of *strong scale effect* differ from the ones we derive for the presence of *weak scale effect*.[11] Nevertheless, our results do reconfirm that the different saving motives are crucial in determining the role of population in R&D-based growth.

The remainder of the paper is organized as follows. Section 2 presents the detailed model. Section 3 studies weak-scale effects with life-cycle saving only. Section 4 introduces the bequest motive for saving. Section 5 summarizes and discusses the results in comparison to the current literature, and Section 6 concludes this study.

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[10] Strulik (2005) find that the sign of the weak scale effect is negative (positive) under the Millian (Benthamite) type of parental preferences (see Theorem 2, p.137 there). Diwakar and Sorek (2016b) elaborate on these results by deriving non-monotonic, hump shape, relation between population growth and innovation under the Beckerian type of altruism, which is abstracted in the aforementioned references.

[11] For example, *DJ* find that the sign of the strong scale effect always depends on the strength of bequest motive, whereas we find that the relevance of bequest-motive strength to the sign of weak scale effect, depends on the *IES*.

2 The Model

We take the variety-expansion model presented in the textbook of Barro and Sala-I-Martin (2004, ch.6)[12], hereafter "BS", and accommodate it to the OLG framework. Hence, the preferences and technologies presented below, and the implied static optimization problems of the firms are identical to those presented in BS. However, unlike BS who study the infinitely living agents, we analyze the OLG demographic setup: each consumer lives for two periods. In the first period she supplies one unit of labor and in the second period she retires. Cohort (generation) size is increasing at an exogenous constant rate n, which is also the growth rate of the labor force and overall population.

2.1 Production and Innovation

The final good Y is produced by perfectly competitive firms with labor and differentiated inputs, to which we refer as "machines"

$$Y_t = AL_t^{1-\alpha} \int_0^{M_t} K_{i,t}^{\alpha} \,\mathrm{d}i \quad \alpha \in (0,1) \tag{1}$$

where A is a productivity factor, L_t and $K_{i,t}$ are labor input and the utilization level of machine i in period t, respectively, and M_t measures the number of available machine varieties[13]. Once invented, the new machine variety is eternally patented. Machines fully depreciate after one usage period, and the final good price is normalized to one. Under symmetric equilibrium, utilization level for all machines is the same, i.e. $K_{i,t} = K_t \ \forall\ i$, and thus total output is

$$Y_t = AM_t {K_t}^{\alpha} L_t^{1-\alpha} \tag{1a}$$

The labor market is perfectly competitive, and therefore the equilibrium wage and aggregate labor income are $w_t = A(1-\alpha)M_t {K_t}^{\alpha} L_t^{-\alpha}$ and $w_t L_t = A(1-\alpha)M_t {K_t}^{\alpha} L_t^{1-\alpha}$, respectively. The profit for the final good producer is $\pi_{i,t} = AL_t^{1-\alpha} \int_0^{M_t} K_{i,t}^{\alpha} \,\mathrm{d}i - \int_{i=1}^{M_t} p_{i,t} K_{i,t} \,\mathrm{d}i - w_t L_t$, where $p_{i,t}$ is the price of input i. Profit maximization yields the demand for each machine: $K_{i,t}^{d} = A^{\frac{1}{1-\alpha}} L_t \left(\frac{\alpha}{p_{i,t}}\right)^{\frac{1}{1-\alpha}}$, for which the periodic surplus from machine i is $PS_{i,t} = \left[p_{i,t} - (1+r_t)\right] K_{i,t}^{d}$. This surplus is maximized by the standard monopolistic price $p_{i,t} = \frac{1+r_t}{\alpha} \forall i, t$[14].

[12] Aghion and Howitt (2009) use the same model in Chapter 3.4 of their textbook.

[13] The elasticity of substitution between different varieties is $\frac{1}{\alpha}$.

[14] BS abstract from the timing of investment, setting the cost of each machines (in terms of output units) to one, and therefore having the optimal monopolistic price $p = \frac{1}{\alpha}$ (equations 6.9-6.10 on pp. 291-292 there). In their continuous time framework this abstraction has no effect on any of the results.

Plugging this price in $K^d_{i,t}$, and then back in (1a), we obtain

$$\begin{aligned} Y_t &= A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r_t}\right)^{\frac{\alpha}{1-\alpha}} M_t L_t \\ y_t &\equiv \frac{Y_t}{L_t} = A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r_t}\right)^{\frac{\alpha}{1-\alpha}} M_t \end{aligned} \tag{1b}$$

Where $y_t \equiv \frac{Y_t}{L_t}$ is per-worker output. Innovation technology follows the specification of *BS* in the analysis of scale effect[15], where the cost of innovating a new variety, denoted η_t, is

$$\eta_t = \eta A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r_t}\right)^{\frac{\alpha}{1-\alpha}} L_t \tag{2}$$

Where $\eta > 0$ is a cost parameter. New and old varieties play equivalent role in production as, reflected in their symmetric presentation in (1). Therefore the market value of old varieties equals the cost of inventing a new one - η_t. As we assume machine-varieties are patented forever, patents are being traded inter-generationally - young buy patents from old. Hence the return on patent ownership - over old and new technologies is $1+r_{t+1} = \frac{PS_{i,t+1}+\eta_{t+1}}{\eta_t}$. Plugging the explicit expressions for the surplus and the innovation cost, we obtain the stationary interest rate[16]:

$$1+r = (1+n)\left[\frac{\alpha(1-\alpha)}{\eta}+1\right], \forall t \tag{3}$$

Following (1b), output growth rate and the per-capita output growth[17], denoted g_Y and g_y respectively, depend on the expansion rate of machine-varieties range, denoted g_M

$$\begin{aligned} 1+g_{Y,t+1} &\equiv \frac{Y_{t+1}}{Y_t} = (1+n)(1+g_{M,t+1}) \\ 1+g_{y,t+1} &\equiv \frac{y_{t+1}}{y_t} = 1+g_{M,t+1} \end{aligned} \tag{4}$$

2.2 Preferences

Lifetime utility, for an agent born in period t, is derived from consumption over two periods, based on the *CIES* instantaneous-utility specification

$$U(c_{t,1}, c_{t,2}) = \frac{c_{t,1}^{1-\theta}}{1-\theta} + \beta\frac{c_{t,2}^{1-\theta}}{1-\theta} \tag{5}$$

[15] See Chapter 6.1.7 on the analysis of scale effect and population growth (p.302 there). Equation (2) implies that variety expansion rate, which defines productivity growth in this model, depends positively on the share of output devoted to R&D. This specification aligns with the empirical regularities summarized in that chapter, which were originally presented by Jones (1995).

[16] Our results would hold if we assume that patents ownership is transferred from parents to offspring, like in the model with infinitely living agents. Then, however, the interest rate would be $1+r = \frac{(1+n)\alpha(1-\alpha)}{\eta}$, which corresponds to the one presented in *BS* (adjusted for continuous time).

[17] Notice that total population and the labor force grow at the same rate, implying equal growth rates for per-worker output and per-capita output.

where $\beta \in (0,1)$ is the subjective discount factor, and $\frac{1}{\theta}$ is the elasticity of inter-temporal substitution. Young agents allocate their labor income between consumption and saving, denoted s. The solution for the standard optimal saving problem is $s_t = \frac{w_t}{1+\beta^{-\frac{1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}$. Hence, aggregate saving is $S_t = \frac{w_t L_t}{1+\beta^{-\frac{1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}$, which after substituting the explicit expressions for w_t becomes

$$S_t = \frac{M_t(1-\alpha)A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}} L_t}{1+\beta^{-\frac{1}{\theta}}(1+r)^{1-\frac{1}{\theta}}} \tag{6}$$

3 Life-Cycle Saving

The saving from labor income in (6) are allocated to three types of investment: buying patents over old varieties, inventing new varieties, and forming specialized machines. Hence aggregate investment in each period, I_t , satisfies

$$I_t = M_{t+1}\left[\eta_t + A^{\frac{1}{1-\alpha}} L_{t+1}\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\right] \tag{7}$$

Notice that higher population growth rate between period t and $t+1$, has direct positive effect on the demand for each machine variety - due to the increase in L. However, following (3), a higher population growth rate also increases the interest rate, which thereby increases machines price and therefore decreases the demand for each machine variety. By equalizing (6) and (7), we impose the equilibrium condition $I_t = S_t$, to obtain the dynamic equation that governs variety expansion rate:

$$\frac{M_{t+1}}{M_t} \equiv 1+g_M = 1+g_y = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}} L_t}{\left[\eta_t + A^{\frac{1}{1-\alpha}} L_{t+1}\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\right]\left[1+\beta^{-\frac{1}{\theta}}(1+r)^{1-\frac{1}{\theta}}\right]} \tag{8}$$

Plugging (2) and (3) in (8) yields

$$1+g_y = \frac{\left(\frac{\alpha(1-\alpha)}{\eta}+1\right)(1-\alpha)}{(\alpha+\eta)\left[1+\beta^{-\frac{1}{\theta}}\left[(1+n)\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]\right]^{1-\frac{1}{\theta}}\right]} \tag{8a}$$

Proposition 1 *With no bequest motive, the effect of population growth on per-capita output growth depends on the sign of* $1-\frac{1}{\theta}$*. For* $IES \equiv \frac{1}{\theta} > 1$ *(*$IES < 1$*) there is postivie (negatvie) weak scale effect, i.e.* $\frac{\partial g_y}{\partial n} > 0$ *(*$\frac{\partial g_y}{\partial n} < 0$*).*

Proof. Proof is by inspection of equation (8a). ■

4 Bequests

We introduce bequest motive for saving that resembles a joy-of-giving in consumers' preferences, similar to *DJ,* which is common to the literature written in the OLG framework (see for example Strulik et al 2013):

$$u(c_t, c_{t+1}, b_t) = \frac{(c_{t,1})^{1-\theta}}{1-\theta} + \beta \left[\frac{(c_{t,2})^{1-\theta}}{1-\theta} + \kappa(1+\varphi(n)) \frac{\left(\frac{b_t}{1+n}\right)^{1-\theta}}{1-\theta} \right] \tag{9}$$

where c_1, c_2 denote consumption when young and old, respectively, and b_t is the total bequest left by a representative parent in period t (hence $\frac{b_t}{1+n}$ denotes per-child bequest). The parameter $\kappa \geq 0$ measures the weight placed on utility from the bequest. Our formulation departs from *DJ* by the term $\varphi(n)$, which captures the potential interaction between the number of kids and the utility from per-child bequest-giving level. At this point, we do not define exact specification for $\varphi(n)$, but we will further discuss it below (following Proposition 2). Each young agent, maximizes her lifetinme utiltiy (9), subject to the budeget constraint: $w_t + \frac{b_{t-1}}{1+n} = c_{t,1} + \frac{c_{t,2}+b_t}{1+r}$. Applying this budget constraint to (9) we write the indirect utility function

$$u(s, w_t, b_{t-1,t}, r) = \frac{(w_t + \frac{b_{t-1}}{1+n} - s_t)^{1-\theta}}{1-\theta} + \beta \left[\frac{[s_t(1+r) - b_t]^{1-\theta}}{1-\theta} + \kappa(1+\varphi(n)) \frac{\left(\frac{b_t}{1+n}\right)^{1-\theta}}{1-\theta} \right] \tag{9a}$$

Differentiating (9a) with respect to s and b we obtain the following first order conditions

$$s_t = \frac{w_t + \frac{b_{t-1}}{1+n}}{\frac{\beta^{-\frac{1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}{1+(1+n)^{\frac{\theta-1}{\theta}}[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}} + 1}, \quad b_t = s_t \frac{1+r}{\frac{(1+n)^{\frac{1-\theta}{\theta}}}{[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}} + 1} \tag{10}$$

Optimal saving is still a fraction of the resources available to the young (worker), which now combine labor income and the bequest she has inherited from her parents. Hence, the operative bequest motive relaxes the former dependency of saving (and thereby investment and innovation rate) on labor income. The fraction of saving out of young's income depends now not only on the interest rate, but also on the parameters of the bequest motive. Stronger bequest motive increases saving, and population growth rate affects saving trough the interest rate as before, but also through the expression $(1+n)^{\frac{\theta-1}{\theta}} [\kappa(1+\varphi(n))]^{\frac{1}{\theta}}$. The effect of first factor in this expression on saving depends on the *IES*. Here, population growth rate works as depreciation rate that erodes per-child bequest level. Hence, its effect is inverse to the effect of the interest rate. This effect has life-cycle saving properties due to the timing of parents utiltiy from bequests giving - during the second period of life. The second factor has positive effect on saving, due to increased marginal utility from per-child bequest. However, it changes with population growth according to the specification of $\varphi(n)$.

The optimal per-child bequest level, is a certain fraction of capital income, $s_t(1+r)$. This fraction is a function of the population growth rate and the bequest motive. As explained above, population growth rate erodes per-child bequest level, and thus work as a negative interest rate. As the utility from bequest takes place during retirement, the effect of lower return bequest on bequest per child depends on the *IES*. The effect of the strength of bequest motive on per-child optimal bequest is positive.

The first condition in (10) implies that aggregate savings is given by

$$S_t = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}} M_t \left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}} L_t + B_{t-1}}{\frac{\beta^{-\frac{1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}{1+(1+n)^{\frac{\theta-1}{\theta}}[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}}+1} \tag{11}$$

Where $B_{t-1} = \frac{L_t b_{t-1}}{1+n}$, is aggregate bequests given to workers who were born in period t. Notice that for $\kappa = 0$ the aggregate saving level defined in (11) falls back to the one presented in (6). The denominator of (11) reveals the way bequest motive interacts with the weak scale effect in determining aggregate saving and thereby economic growth: the term $(1+n)^{\frac{\theta-1}{\theta}}$ marks the diluting effect of n on per-child bequest level. The sign of this effect on saving, just like the interest rate, depends on θ, i.e. the *IES*. The term $[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}$ marks the strength of the saving motive for per-child bequest-giving. The impact of n on this term depends on the sign of $\varphi'(n)$.

The second condition in (10) implies that $B_{t-1} = \frac{1+r}{\left(\frac{(1+n)^{\frac{1-\theta}{\theta}}}{[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}}+1\right)} S_{t-1}$, and the equilibrium condition $S_{t-1} = I_{t-1}$ requires

$$B_{t-1} = \frac{1+r}{\left(\frac{(1+n)^{\frac{1-\theta}{\theta}}}{[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}}+1\right)} M_t \left(\eta_{t-1} + A^{\frac{1}{1-\alpha}} L_t \left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\right) \tag{12}$$

Substituting the latter expression along with (3) back into (11) and equalizing to (7), i.e. setting $S_t = I_t$, we obtain

$$\frac{M_{t+1}}{M_t} = 1 + g_y = \frac{\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]\left[\frac{(1-\alpha)}{\alpha+\eta}\left[(1+n)^{\frac{1-\theta}{\theta}} + [\kappa(1+\varphi(n))]^{\frac{1}{\theta}}\right] + [\kappa(1+\varphi(n))]^{\frac{1}{\theta}}\right]}{\beta^{-\frac{1}{\theta}}\left(\frac{\alpha(1-\alpha)}{\eta}+1\right)^{\frac{\theta-1}{\theta}} + \left[(1+n)^{\frac{1-\theta}{\theta}} + [\kappa(1+\varphi(n))]^{\frac{1}{\theta}}\right]} \tag{13}$$

Notice that, for $\kappa = 0$ equation (12) boils down to the growth rate presented in equation (8a). Rearranging (12) yields

$$1 + g_y = \frac{\left[\frac{(1-\alpha)}{\alpha+\eta} + \frac{1}{1+\psi}\right]\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]}{\frac{\beta^{-\frac{1}{\theta}}(1+n)^{\frac{\theta-1}{\theta}}\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]^{\frac{\theta-1}{\theta}}}{1+\psi^{-1}}+1} \tag{12a}$$

Where $\psi \equiv \frac{(1+n)^{\frac{1-\theta}{\theta}}}{[\kappa(1+\varphi(n))]^{\frac{1}{\theta}}}$. The growth rate defined in (12a) presents a complex impact of the population growth rate, which works through the bequest motives (captured in ψ) and the interest rate effect presented in the denominator of (12a). The sign of the interest-rate effect depends solely on the IES, i.e. θ, as defined in Proposition 1. The sign of the bequest motive effect, i.e. the sign of $\frac{\partial\psi}{\partial n}$, depends on the sign of $(1-\theta)(1+n)^{-1} - \varphi'(n)(1+\varphi(n))^{-1}$, which is a function of n. Hence, (12a) implies ambiguous effect of population growth on per-capita output growth, which may be non-monotonic. To further characterize the relation between population growth and output growth, we focus first on the case $\theta = 1$ for which (12) becomes

$$1+g_y = \frac{1}{\frac{1}{\beta[1+\kappa(1+\varphi(n))]}+1}\left[\frac{(1-\alpha)}{\alpha+\eta} + \frac{1}{\frac{1}{\kappa(1+\varphi(n))}+1}\right]\left[\frac{\alpha(1-\alpha)}{\eta}+1\right] \tag{14}$$

Proposition 2 *In the presence of bequest saving-motive, for $\theta = 1$ the effect of population growth on per-capita income growth, $\frac{\partial g_y}{\partial n}$, depends solely on the sign of $\varphi'(n)$.*

Proof. Differentiating (13) for n reveals that the sign of $\frac{\partial g_y}{\partial n}$ is given by the sign of $-\varphi'(n)(1+\varphi(n))^{-1}$. ■

Hence, if $\theta = 1$ and the parent cares about per-child bequest only, i.e. $\varphi(n) = 0$, there is no weak scale effect: $\frac{\partial g_y}{\partial n} = 0$. These parental preferences are in line with the Millian preference specification employed in *BS*. For the case $\varphi'(n) > 0$ (< 0), which is in line with the Benthamite ("Beckerian"[18]) specification, Proposition 2 implies positive (negative) weak scale effect.

We turn now to further explore the case where parents care about per-child bequest giving $\varphi(n) = 0$, for which equation (12) becomes

$$\frac{M_{t+1}}{M_t} = \frac{\left[\frac{1-\alpha}{\alpha+\eta}\left(\kappa^{-\frac{1}{\theta}}(1+n)^{\frac{1-\theta}{\theta}}+1\right)+1\right]\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]}{\beta^{-\frac{1}{\theta}}\kappa^{-\frac{1}{\theta}}\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]^{\frac{\theta-1}{\theta}}+\kappa^{-\frac{1}{\theta}}(1+n)^{\frac{1-\theta}{\theta}}+1} \tag{15}$$

Proposition 3 *For $\varphi'(n) = 0$, the effect of population growth on per-capita output growth, $\frac{\partial g_y}{\partial n}$, is positive (negative) for $\theta < 1$ $(\theta > 1)$ and sufficiently weak (strong) bequest motive.*

Proof. Differentiating (14) for n reveals that, for $\theta < 1$ $(\theta > 1)$, $\frac{\partial g_y}{\partial n} > 0$ *iff* $\beta^{-1}\left(\frac{1-\alpha}{\alpha+\eta}\right)^{\theta}\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]^{\theta-1} > \kappa$ ($\beta^{-1}\left(\frac{1-\alpha}{\alpha+\eta}\right)^{\theta}\left[\frac{\alpha(1-\alpha)}{\eta}+1\right]^{\theta-1} < \kappa$). for $\theta = 1$, the sign of $\frac{\partial g_y}{\partial n}$ is independent of κ as already stated in Proposition 2. ■

[18] Presented in Barro and Becker (1988) and Becker et al.(1990).

5 Discussion

We turn now to summarize our results, in compariosn with BS reference model of infinitely living agents and *DJ*'s analysis of strong scale effect in the OLG framework with AK technology. Table 1 summarises the results we have obtained in Propositions 1-3.

Table 1: Results Summary

	$\theta=1$	$\theta>1$	$\theta<1$
$\kappa=0$	0	-	+
$\varphi'(n)=0$	0	for large κ : -	for small κ : +
$\varphi'(n)>0$	+	NA	NA
$\varphi'(n)<0$	-	NA	NA

The first row in Table 1 present the result obtained in Proposition 1. The other three rows are for $\kappa > 0$. The first Column in Table 2 presents the results obtained in Proposition 2, and the second row presents the results obtained in Proposition 3.

The corresponding model of infinitely living agents, presented in *BS*, follows that Millian preferences - households maximize per-capita utility of their dynasty member. Hence, aggregate consumption growth follows the standard Euler equation[19]: $\frac{\dot{C}}{C} = \frac{1}{\theta}(r-\beta)$, and per-capita consumption follows $\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\beta-n)$ where the interest rate is given by[20] $r = n + \frac{\alpha(1-\alpha)}{\eta}$. Combining the two latter conditions yields the stationary growth rates for aggregate and per-capita consumption, which apply also for aggregate and per-capita income:

$$g_{C,Y} = \frac{1}{\theta}\left[n + \frac{\alpha(1-\alpha)}{\eta} - \beta\right], \quad g_{c,y} = \frac{1}{\theta}\left[\frac{\alpha(1-\alpha)}{\eta} - \beta\right] \tag{BS.1}$$

Modifying *BS*'s model for the Benthamine preferences, where households maximizing aggregate dynastic lifetime utility, yields the following Euler conditions: $\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\beta)$ and the corresponding growth expressions:

$$g_{c,Y} = \frac{1}{\theta}\left[n + \frac{\alpha(1-\alpha)}{\eta} - \beta\right] \tag{BS.2}$$

Equations (BS.1)-(BS.2) show that in the economy of infinitely living homogeneous agents, which abstracts the life-cycle saving motive, the *IES* plays no role in the presence (or sign) of

[19] Equation (6.22) on p.295 there, in which the parameter ρ the time preference parameter (denoted here as β).
[20] Equation (6.35) on p. 302 there.

the weak scale effect. Under the Millian preferences, there is no weak scale effect and under the Benthamine preferences, there is a positive weak scale effect. These results are consistent with the ones presented in cells second and thirds cells of column 1, for which the life-cycle saving motive is muted and $\varphi'(n) = 0$ and $\varphi'(n) > 0$, respectively. Nevertheless, the first row of Table 1 show that the life-cycle motive by itself affects the sign of the weak scale effect. The second and third cells of the second row in Table 1 show that when this effect of life-cycle saving is operative, the overall relation between population growth and economic growth becomes more complicated, and it interacts with the strength of the bequest motive, captured by the parameter κ (which represent the relative utility from the joy-of-giving vs. from parent's own consumption).

In reference to the results obtained by *DJ*, it is worthwhile noting that they find strong scale-effect for $\theta = 1$ under the technological parameters used in our model[21]. Furthermore, *DJ* find that sufficiently *strong* bequest motive, i.e. sufficiently large κ, is necessary for the presence of strong scale-effect in their model for any *CES* (Constant Elasticity of Substitution between labor and capital) production technology (where the bequest motive is defined solely by the parameter κ). These findings are not in line with our first and second propositions, which imply that for $\theta = 1$ the sign of the weak scale-effect does not depend on the value of κ but on the type of parental preference for bequests, reflected in the sign of $\varphi'(n)$.

6 Conclusions

In this study, we have shown how the two alternative saving motives - life-cycle considerations and intergenerational bequests - determine the relation between population growth and economic prosperity. First, we showed that in the standard OLG economy, where life-cycle considerations are the sole saving motive, the effect of population growth on economic growth depends on the *IES*, whereas in our reference textbook model of infinitely living agents there is no weak scale effect for any value of the *IES*.

Then, we showed that, when the life-cycle saving impact is neutralized, the sign of the weak scale-effect depends on the specification of the parental preference for bequests, in a way that aligns with the findings of the literature on infinity living agents.

Finally, acknowledging that both modeling approaches are unrealistically extreme, we also analyzed a hybrid model with both bequest and life-cycle saving motives. In this case the relation between population growth and economic growth is complex in general and may be non-monotonic, as it depends on the exact specification of the bequest motive and its relative strength.

Hence, we conclude that the counterfactual weak scale-effect presented in the second-generation models of R&D-Based growth interacts with the assumed demographic structure and its implies saving motives. Therefore, this paper contributes to the recent line of research aim to align modern growth theory with the empirical findings regarding ambiguous and possibly non-monotonic relation between population growth and innovation.

[21] See Corollary 2 on p. 1643 there for $\sigma = 1$ (by their notation), which is the elasticity of substitution between labor and capital in our model.

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