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Abstract

We study the implications of patents breadth and duration (length) to growth and welfare, in an overlapping generations economy of finitely living agents. This demographic structure gives room to inter-generational trade in old patents and life-cycle saving motive, which prove to be relevant to the implications of different patent-policy dimensions. Patent breadth protection affects life-cycle saving and thereby aggregate investment, and the allocation investment between patents and physical capital. Patents duration affects also the stock of, and trade in, old patents. We show that, these unique characteristics of the OLG economy provide a case for incomplete patent breadth protection and finite patent duration, which are both growth and welfare enhancing. Furthermore, we show that the implications of patent policy to growth depend on whether the differentiated inputs are intermediate goods or investment goods. Our results contrast with the ones derived in previous studies that employed the same technologies and preferences in model economies of infinitely living agents. Hence, this work highlights the importance of the assumed demographic structure to the implications of patent policy.

JEL Classification: : L16, O-30

Key-words: IPR, Patent length, Patent Breadth, Growth, OLG

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1 Introduction

There is, by now, a large literature on the role of patent policy in modern growth theory, namely, the effects of patents strength on R&D-based growth and welfare. This literature, however, was almost exclusively written in model-economies with infinitely living agents. This study contributes a comparative analysis on the implications of patent policy to growth and welfare, for the alternative workhorse framework for macroeconomic modeling - the overlapping generations (OLG) economy of finitely living agents.

We highlight two unique implications of the OLG demographic structure to patent policy, in a variety expansion model (i.e. horizontal differentiation). First, we show that in the OLG economy patent breadth and patent duration (length) have uneven effects on growth. Secondly, we show that the effect of patent policy on growth depends on whether the differentiated inputs are defined as intermediate goods (see for example Kwan and Lai 2003, Barro and Sala-i-Martin 2004, Cysne and Turchick 2012 and Zeng et al. 2013), or as investment goods - i.e. physical capital, or "machines" (see for example Rivera-Batiz and Romer 1991, Strulik et al. 2013, and Prettner 2014). These two types of inputs differ in their formation timing: the intermediate goods are formed in the same period they are being used, whereas the investment goods must be formed one period ahead of utilization.

The demographic structure of the OLG implies that, if patent duration exceeds the lifetime of patent holders, inter-generational trade in old patent prevails. This trade in old patents crowds out investment in new innovative R&D. Moreover, in the OLG economy aggregate savings rely on labor income of young workers. When the differentiated inputs are intermediate goods, all savings out of labor income are invested in old and new patents. Hence shortening patent length enhances innovation, and thereby growth, by reducing the investment in the (smaller) stock of old patents. However, the effect of patent breadth growth is positive because it's overall effect on output, and thereby labor income and saving and growth, is positive. Hence, we show that when the differentiated inputs are intermediate goods, growth is maximized with minimal patent length and maximal patent breadth protection.

The above result contrasts with the those obtained by Kwan and Lai (2003), Cysne and Turchick (2012) and Zeng et al. (2013), who studied the same variety expansion growth model we employ here[1] with infinitely living agents. These three studies differ only in their modelling approach of patent policy. The first two studies model patent policy through constant imitation rate, following Helpman (1993)[2]. This modeling approach can be interpreted as stochastic patent duration, as will be explained below. The third paper models patent policy along two (more natural) dimensions: deterministic patent duration along with price regulation which is equivalent to our modeling of patent breadth. All three studies conclude that growth is maximized under complete patent protection.

[1] That is, the same innovation and production technologies, and instantaneous preferences.

[2] Cysne and Turchick (2012) corrected Kwan and Lai's (2003) early analysis and modified their results accordingly. The same modeling approach and conclusions are presented in Barro and Sala-I-Martin's (2004) textbook (Ch. 6).

Chou and Shy (1993) where the first to identify the crowding-out effect of long patent-duration due to trade in old patent, which hinders innovation and growth. With technologies and instantaneous preferences that differ from the ones we employ here, they also find that one period patent duration yields higher growth rate than infinite patent duration. In their model, differentiated consumption goods are produced with labor and the time of utilization, hence it corresponds to production with intermediate goods. However, they did not consider patent breadth protection. Sorek (2011) studies the effect of patent breadth and patent length protection on quality growth (i.e. vertical differentiation) in an OLG economy, where differentiated consumption goods are also produced with labor. In his setup the effect of patent policy depends solely on the elasticity of inter-temporal substitution (through the effect of the interest rate life-cycle saving in the OLG model), unlike in the present work.

Next, we study the counterpart model economy that produces with differentiated investment goods, that are formed one period ahead of use. Here, the differentiated machines compete with patents over the allocation of saving as an alternative form of investment. As a result, weakening patent breadth protection induces contradicting effects on growth. Weakening patent breadth protection lowers the price of patented machines, which in turn increases demand for machines. With more machines being utilized output and labor income are higher, and therefore aggregate saving an investment is increasing[3]. This is the positive static effect of loosening patent breadth protection on growth. However, on the other hand, higher demand for machine shifts investment away from patents towards physical capital. This is the negative dynamic effect of weakening patent breadth protection on growth. Shortening patent duration induces the same trade-off through lowering the average price of machine varieties, plus additional positive effect on growth due to decreasing the investment in old patent (i.e. the crowding-out effect).

We will show that when differentiated inputs are investment goods growth is always maximized with incomplete patent breadth protection or intermediate patent length (but not always with minimal patent length), and that incomplete patent breadth protection is always welfare maximizing. By contrast, in the counterpart models of infinitely living agents (cited above) the implications of patent policy does not depend on the timing of differentiated-inputs formation. There, the effect of patent policy on growth works solely through the effect of patent protection on the rate of return (which is definitely positive there as well as in our OLG economy), which in turn determines the growth rate through the standard Euler equation. As mentioned above, this study finds that growth is always maximized with complete protection but welfare may be maximized with incomplete protection.

The paper proceeds as follows: Section 2 presents the model. Section 3 studies the implications of patent policy to growth for differentiated intermediate goods, and Section 4 studies it implications to growth and welfare for differentiated investment goods. Section 5 concludes this study.

[3] This is in line with Jones and Manuelli (1992) who showed that in neoclassical economy of finitely living agents, growth is bounded by the ability of the young generation to purchase capital held by the old. One of the remedies they consider to support sustained growth in such economy is direct income transfers from old to young. Similarly, Uhlig and Yanagawa (1996) showed that reliance on capital-income taxation can enhance growth.

2 Model

The model presented below is identical to the one employed by Kwan and Lai (2003), Cysne and Turchick (2012) and Zeng et al. (2013) and Barro and Sala-I-Martin (2004, Ch. 6), except for its demographic structure that follows Diamond's (1965) canonical OLG framework: a population of constant size composes two overlapping generations, of measure L, in each and every period - "young" and "old". Each agent is endowed with one unit of labor to be supplied inelastically when young. Old agents retire to consume their saving.

We present the model here under the assumption of full patent protection - i.e. infinite patent duration and complete patent breadth protection that implies innovator can charge unconstraint monopolistic price. We study the implications of incomplete patent protection in Section 3.

2.1 Differentiated inputs as intermediate goods

The final good Y is produced by perfectly competitive firms with labor and differentiated intermediate goods, to which we refer as "machines"

$$Y_t = AL_t^{1-\alpha} \int_0^{N_t} K_{i,t}^{\alpha} \, \mathrm{d}i \quad \alpha \in (0,1) \tag{1}$$

where A is a productivity factor, L is the constant labor supply, $K_{i,t}$ is the utilization level of machine-variety i in period t, respectively, and N_t measures the number of available machine-varieties[4]. In the benchmark case, once invented, each machines variety is eternally patented. Machines fully depreciate after one usage period, and the final good price is normalized to one. Under symmetric equilibrium, utilization level for all machines is the same, i.e. $K_{i,t} = K_t \; \forall \; i$, and thus total output is

$$Y_t = AN_t {K_t}^{\alpha} L^{1-\alpha} \tag{1a}$$

The labor market is perfectly competitive, and therefore the equilibrium wage and aggregate labor income are $w_t = A(1-\alpha)N_t {K_t}^{\alpha} L^{-\alpha}$ and $w_t L = A(1-\alpha)N_t {K_t}^{\alpha} L^{1-\alpha}$, respectively. The profit for the final good producer is

$$\pi_{i,t} = AL_t^{1-\alpha} \int_0^{N_t} K_{i,t}^{\alpha} \, \mathrm{d}i - \int_{i=1}^{N_t} p_{i,t} K_{i,t} \, \mathrm{d}i - w_t L_{t,i}$$

where $p_{i,t}$ is the price of intermediate good i in period t, and $L_{i,t}$ is labor input employed in the production of this good. Profit maximization yields the demand for each machine: $K_{i,t}^d = A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha}{p_{i,t}}\right)^{\frac{1}{1-\alpha}}$. Given this demand, the periodic surplus for the patented machine i is $PS_{i,t} = (p_{i,t} - 1) K_{i,t}^d$, which is maximized by the standard monopolistic price $p_{i,t} = \frac{1}{\alpha} \forall i, t$. Plugging this price into $K_{i,t}^d$ yields $K_{i,t}^d = A^{\frac{1}{1-\alpha}} L \alpha^{\frac{2}{1-\alpha}} \; \forall i, t$. Then, plugging the latter expression back into (1a)

[4] The elasticity of substitution between different varieties is $\frac{1}{\alpha}$.

yields the following expression for total output

$$Y_t = A^{\frac{1}{1-\alpha}} \alpha^{\frac{2\alpha}{1-\alpha}} N_t L \tag{1b}$$

Following (1b), output growth rate (which coincides with per-capita output growth)[5], denoted $g_Y \equiv \frac{Y_{t+1}}{Y_t}$, coincides with the rate of machine-varieties expansion, denoted $g_{N,t+1} \equiv \frac{N_{t+1}}{N_t}$, i.e. $g_Y = g_N \equiv g$.

Innovation technology follows lab-equipment specification, so the cost of a new blue print is η. New and old varieties play equivalent role in production as reflected by their symmetric presentation in (1). Therefore, the market value of old varieties equals the cost of inventing a new one - η. When machine-varieties are patented forever, all old patents are being traded, as the young saver buy patents from the old retirees. Hence the return on patent ownership - over old and new technologies is $1 + r_{t+1} = \frac{PS_{i,t+1}+\eta}{\eta}$. Plugging the explicit expressions for the surplus and the innovation cost we obtain the stationary interest rate

$$1 + r = \frac{\left(\frac{1}{\alpha} - 1\right) K^d + \eta}{\eta} = \frac{\left(\frac{1}{\alpha} - 1\right) A^{\frac{1}{1-\alpha}} L \alpha^{\frac{2}{1-\alpha}} + \eta}{\eta} \tag{2}$$

2.2 Preferences

Lifetime utility of the representative agent, who was born in period t, is derived from consumption over two periods based on the logarithmic instantaneous-utility specification[6]

$$U_t = \ln c_{1,t} + \rho \ln c_{2,t} \tag{3}$$

where $\rho \in (0, 1)$ is the subjective discount factor. Young agents allocate their labor income between consumption and saving, denoted s. The solution for the standard optimal saving problem is $s_t = \frac{w_t}{1+\rho^{-1}}$. Hence, aggregate saving is $S_t = \frac{w_t L}{1+\rho^{-1}}$, which after substituting the explicit expressions for w_t becomes

$$S_t = \frac{N_t(1-\alpha)A^{\frac{1}{1-\alpha}} L}{1+\rho^{-1}} \left[\left(\alpha^2\right)^{\frac{\alpha}{1-\alpha}} - \left(\alpha^2\right)^{\frac{1}{1-\alpha}} \right] \tag{4}$$

3 Patent Policy and Growth

We first solve for the stationary growth rate under the assumed complete patent protection. In each and every period, aggregate saving is allocated to investment in old patents and investment in new patents (i.e. invention of new machine varieties)

[5] As both total population and the labor force are constant.

[6] It is well known that under the assumed demographic structure, the logarithmic instantaneous utility implies that saving (and investment) level is independent of the interest rate. In section (5) we will consider the implications of the general CRRA preference form.

$$I_t = N_{t+1}\eta \tag{5}$$

Imposing the equilibrium condition $S = I$, we equalize (4) and (5), to derive the stationary rate of variety expansion (and output growth)

$$1 + g_y = \frac{N_{t+1}}{N_t} = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}L}{(1+\rho^{-1})\,\eta}\left[\left(\alpha^2\right)^{\frac{\alpha}{1-\alpha}} - \left(\alpha^2\right)^{\frac{1}{1-\alpha}}\right] \tag{6}$$

Equation (6) implies that growth is subject to strong scale effect, i.e. it increases in labor force size. It increases also in total productivity factor, labor income share, and the discount factor, and decreases in innovation cost.

3.1 Patent breadth

We model patent breadth protection with the parameter λ, which limits the ability of patent holders to charge the unconstraint optimal monopolistic price: $p(\lambda) = \lambda p^* = \frac{\lambda}{\alpha}$ where $\lambda \in (\alpha, 1)$, and thus $p(\lambda) \in \left(1, \frac{1}{\alpha}\right)$. One can think of $p(\lambda)$ as the maximal price a patent holder can set and still deter competition by imitators. Weaker breadth protection lowers the cost of imitation, thereby imposing a lower deterrence price on patent holders[7]. When $\lambda = 1$, patent breadth protection is complete and patent holders can charge the unconstraint monopolistic price $p = \frac{1}{\alpha}$. With zero protection $\lambda = \alpha$, patent holders are losing their market power completely, therefore selling at marginal cost price. Note that as patent breadth protection is weakened machines' price is reduced and therefore demand for each machine-variety is increasing. However, loosening patent breadth protection does not affect directly the inter-generational trade in old patents. Plugging $p(\lambda) = \frac{\lambda}{\alpha}$ in (4) and (5), and imposing $S = I$, we obtain the explicit growth rate as a function of patent breadth protection

$$1 + g_y = \frac{N_{t+1}}{N_t} = \frac{(1-\alpha)A^{\frac{1}{1-\alpha}}L}{(1+\rho^{-1})\,\eta}\left[\left(\frac{\alpha^2}{\lambda}\right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha^2}{\lambda}\right)^{\frac{1}{1-\alpha}}\right] \tag{7}$$

Proposition 1 *Under production with intermediate goods, growth is maximized with maximal patent breadth protection, i.e.* $\lambda^* = 1$.

Proof. Differentiating the right hand side of (7) for λ reveals that growth rate is increasing in patent breadth protection, that is $\forall \lambda \in (\alpha, 1) : \frac{\partial g}{\partial \lambda} > 0$. ■

3.2 Patent length

3.2.1 Deterministic Patent Length

Here we consider deterministic patents' duration, which is a discrete number of periods denoted T, and derive analytical results for the case of one-period duration $T = 1$. Under one-period patent

[7] Our modeling approach is equivalent to the explicit price (ceiling) regulation considered by Zeng et al.(2013).

duration, there is no inter-generational trade in old patents, and only new technologies are sold at the monopolistic price $p_m = \frac{1}{\alpha}$. All old varieties are sold under perfect competition at marginal cost price $p_c = 1$. Hence, demand for machines in each competitive industry is $K^d_{i,t} = A^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}L$. The share of monopolistic industries, denoted μ, depends on the (stationary) growth rate: $\mu = \frac{g}{1+g}$. Hence total output that was given in (1a) modifies to

$$Y_t = A^{\frac{1}{1-\alpha}}N_tL\left[\frac{g}{1+g}\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) + \frac{1}{1+g}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\right] \tag{8}$$

The saving and investment expressions, from equations (4)-(5) also modify accordingly

$$\begin{aligned} S_t &= \frac{(1-\alpha)}{1+\rho^{-1}}N_tA^{\frac{1}{1-\alpha}}L\left[\frac{g}{1+g}\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) + \frac{1}{1+g}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\right] \\ I_t &= N_{t+1}\left(\frac{g}{1+g}\right)\eta \end{aligned} \tag{9}$$

and the equilibrium condition $S_t = I_t$ yields an implicit expression for the stationary growth rate

$$1+g = \frac{(1-\alpha)}{1+\rho^{-1}}\frac{A^{\frac{1}{1-\alpha}}L}{\eta}\left[\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) + \frac{1}{g}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\right] \tag{10}$$

Proposition 2 *For production with intermediate goods, growth rate under one period patent duration is always positive and higher than under infinite patent duration.*

Proof. The left hand side of (10) is increasing linearly in g. Its right hand side is decreasing monotonically in g from infinity, for $g = 0$, to a positive finite value, for $g \to \infty$. Hence, by the fix-point theorem, there exists unique solution for (10) with $g > 0$. For $g \to \infty$, the right hand side of (10) approaches $\frac{(1-\alpha)}{1+\rho^{-1}}\frac{A^{\frac{1}{1-\alpha}}L}{\eta}\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right)$, which is, by (6), the growth rate under infinite patent duration. Hence, the stationary growth rate defined by (10) for one-period patent duration is necessarily higher than the one defined in (6) for infinite patent duration. ∎

Proposition 2 contrasts with the result obtain by Zeng et al. (2013), who find that infinite patent duration maximizes growth the counterpart economy with infinitely living agents. It aligns, however, with the result obtained in Chou and Shy (1993), for OLG economy that produces with differentiated intermediate goods, though with labor as sole production input.

Under any intermediate patent length protection T, patented technologies are being traded at different prices- according to their remaining life time, subject to the following no-arbitrage condition:

$$1+r = \frac{\left(\frac{\lambda}{\alpha}-1\right)K^d_m + v_1}{\eta} = \frac{\left(\frac{\lambda}{\alpha}-1\right)K^d_m + v_2}{v_1} = ... = \frac{\left(\frac{\lambda}{\alpha}-1\right)K^d_m + v_{T-1}}{v_{T-2}} = \frac{\left(\frac{\lambda}{\alpha}-1\right)K^d_m}{v_{T-1}}$$

where v_1 denotes the market value of each technology invented in period 0, when sold in period 1, and v_2 is the market value of those technologies in period 2, and so on and forth. At period

$T-1$, which is one period before the patent expires, the value of the technology goes down to the present value of the surplus, as it will not be sold again in the next period. Longer patent duration increases the equilibrium interest rate. An equivalent (positive) effect of patent duration on the interest rate generates the positive relation between patent duration and growth (through the standard Euler equation) in Zeng et al. (2013), for infinitely living agents economy.

The average market value of all traded technologies, denoted $\overline{v}$ is increasing with patent length protection. For a given patent length protection, T, the stationary growth rate is given by the implicit expression[8]

$$1+g=\frac{1}{1+\rho^{-1}}\frac{(1-\alpha)A^{\frac{1}{1-\alpha}}L}{\overline{v}(T)}\left[(1-\frac{1}{(1+g)^{T}})\left[\left(\frac{\alpha^2}{\lambda}\right)^{\frac{\alpha}{1-\alpha}}-\left(\frac{\alpha^2}{\lambda}\right)^{\frac{1}{1-\alpha}}\right]+\frac{1}{(1+g)^{T}}\left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\right] \tag{10a}$$

Where $\overline{v}(T)$ is the average value of all existing technologies: $\overline{v}(T)=\frac{g}{1+g}\eta+\frac{g}{(1+g)^2}v_1+\frac{g}{(1+g)^3}v_2+ .....\frac{g}{(1+g)^{T+1}}v_T$. However, we find the analysis of this expression non-tractable. We could not also derive a comparison between the stationary growth rates under incomplete patent breadth and finite patent duration, presented in equations (7) and (10).

3.2.2 Stochastic patent length

Here, we follow the patent-policy modeling of Kwan and Lai (2003) and Rubens and Turchick (2012), assuming that each period a fraction $1-\pi$ of the existing patents expire[9]. Accordingly, only the fraction π of new-technologies is effectively patented in the first period of utilization. This means that actual patent-protection length, denoted T, is stochastic, with the expected value $E(T)=\frac{\pi}{1-\pi}$ for all new and old patented technologies. Under this specification the stationary fraction of patented industries, μ, is

$$\mu=\frac{\pi g}{1+g-\pi}\Rightarrow 1-\mu=\frac{(1+g)(1-\pi)}{1+g-\pi} \tag{11}$$

Demand for machines in each competitive industry is $K_{i,t}^{d}=A^{\frac{1}{1-\alpha}}\alpha^{\frac{1}{1-\alpha}}L$ and thus, total output that was given in (1a) modifies to

$$Y_t=A^{\frac{1}{1-\alpha}}N_tL\left[\frac{\pi g}{1+g-\pi}\left(\alpha^{\frac{2\alpha}{1-\alpha}}-\alpha^{\frac{2}{1-\alpha}}\right)+\frac{(1+g)(1-\pi)}{1+g-\pi}\left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\right] \tag{12}$$

Aggregate saving and investment from (6) and (7), also modify accordingly:

$$S_t=\frac{(1-\alpha)}{1+\rho^{-1}}A^{\frac{1}{1-\alpha}}N_tL\left[\frac{\pi g}{1+g-\pi}\left(\alpha^{\frac{2\alpha}{1-\alpha}}-\alpha^{\frac{2}{1-\alpha}}\right)+\frac{(1+g)(1-\pi)}{1+g-\pi}\left(\alpha^{\frac{\alpha}{1-\alpha}}-\alpha^{\frac{1}{1-\alpha}}\right)\right] \tag{13}$$

[8] Note that for $T=1$ the expression in (10a) coincides with (10) and for $T\to\infty$ it coincides with (6).

[9] Or, as in their original interpretation, a fraction π of the patented technologies is being imitated due to lack of patent-protection enforcement.

$$I_t = \left(N_{t+1} - N_t + \frac{\pi g}{1+g-\pi} N_t\right)\eta = \left(\frac{g}{1+g-\pi}\right)\eta N_{t+1} \tag{13a}$$

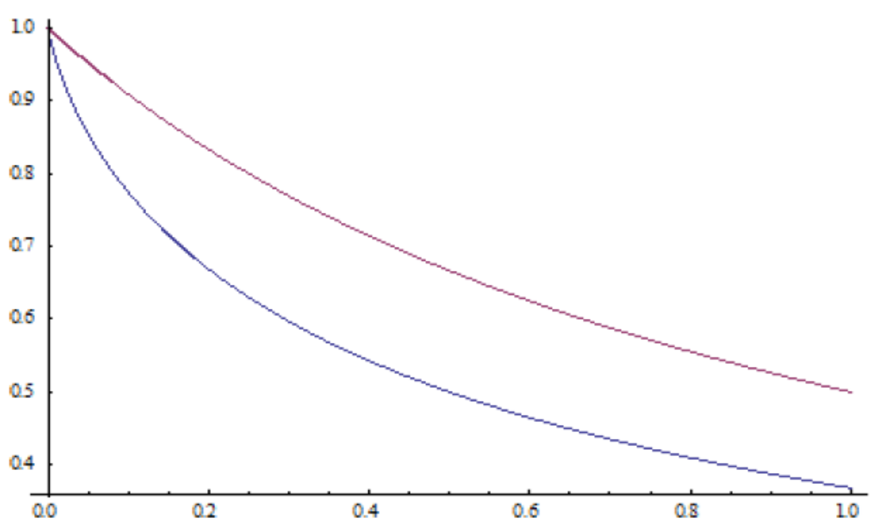
Equalizing aggregate saving from (13) and aggregate investment (13a) yields the following implicit function of the stationary growth rate

$$1+g = \frac{(1-\alpha)}{1+\rho^{-1}} \frac{A^{\frac{1}{1-\alpha}} L}{\eta} \left[\pi \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) + \frac{(1+g)(1-\pi)}{g} \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)\right] \tag{14}$$

Proposition 3 *The growth maximizing stochastic patent duration is zero.*

Proof. Differentiating the right hand side of (14) for π, yields the following first order condition for $\frac{\partial g}{\partial \pi} < 0 : \frac{\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}}{\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}} < \frac{(1+g)}{g}$. This condition necessarily holds if $\frac{\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}}{\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}} < 1$. The latter condition can be written as $\alpha^{\frac{\alpha}{1-\alpha}} < \frac{1}{1+\alpha}$, which holds for any $\alpha \in (0,1)$ as demonstrated in figure 1. ■

Figure 1: Proving Proposition 3



The horizontal axis in figure one take values of α, from zero to one. The blue curve is the function $\frac{1}{1+\alpha}$ and the red one is $\alpha^{\frac{\alpha}{1-\alpha}}$.

4 Differentiated inputs as investment goods

Considering the differentiated inputs as physical-capital (i.e. "specialized machines"), implies that they should be formed one period ahead of utilization, as part of investment uses. We still assume that the formation of one specialized machine takes one unit of input. Therefore the usage cost of each machine, in current value terms, equals the (gross) interest rate $1+r_t$ paid at period t[10]. The current value of the periodic surplus from renting each patented machine, modifies accordingly to $PS_{i,t} = [p_{i,t} - (1+r_t)]\, K_{i,t}^d$, and under complete patent protection this surplus is maximized by the standard monopolistic price $p_{i,t} = \frac{1}{\alpha}(1+r_t)\ \forall i$. Plugging this price into $K_{i,t}^d$ yields $K_{i,t}^d = A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{1+r_t}\right)^{\frac{1}{1-\alpha}}\ \forall i,t$.

Under infinite patent duration, the rate of return on investment in patents is given by $1+r_t = \frac{PS+\eta}{\eta}$. Plugging the explicit term of the surplus yields an implicit expression for the equilibrium interest, which equalizes the return on investment in patents and investment in physical capital:

$$\forall t : 1+r_t = \frac{[p_{i,t} - (1+r_t)]\, K_{i,t}^d + \eta}{\eta} = \frac{(1+r_t)^{-\frac{\alpha}{1-\alpha}} \left(\frac{\lambda}{\alpha} - 1\right) A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda}\right)^{\frac{1}{1-\alpha}} + \eta}{\eta} \tag{15}$$

Lemma 1 *There exists a stationary interest rate that is increasing with patent breadth protection*

Proof. The left hand side of (15) is increasing linearly in r , from one (for $r=0$) to infinity. The right hand side of (15) is decreasing in r from $\frac{\left(\frac{\lambda}{\alpha}-1\right)A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda}\right)^{\frac{1}{1-\alpha}} + \eta}{\eta} > 1$ (for $r=0$) to zero (for $r \to \infty$). Hence, by the fix point theorem there exists positive stationary interest rate r^* that solves (15). Differentiating the right hand side for λ yields a positive derivative for $\lambda < 1$. Hence the value of r^*, which solves (15), is increasing with patent breadth protection λ. ■

Lemma 1 implies that the expression $\lambda\,[1+r(\lambda)]$ in (15) is also increasing in the strength of patent breadth protection λ.

Under one period patent duration there is no inter-generational trade in old patents, implying $1+r_t = \frac{PS}{\eta}$. Therefore, under one period patent duration, with complete equation (15) modifies to

$$\forall t : (1+r_t) = \frac{(1+r_t)^{-\frac{\alpha}{1-\alpha}} \left(\frac{\lambda}{\alpha} - 1\right) A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda}\right)^{\frac{1}{1-\alpha}}}{\eta} \Rightarrow (1+r_t) = A\alpha^2 \left[\frac{\left(\frac{1}{\alpha} - 1\right) L}{\eta}\right]^{1-\alpha} \tag{15a}$$

Clearly, for a given patent breadth protection, the interest rate under one period patent duration in (15a) is lower than under infinite patent duration, as given in (15).

4.1 Patent Breadth

Here we consider the effect of patent breadth protection on growth, under infinite patent protection. Hence, relevant interest rate is the one defined in (15). Plugging $K_{i,t}^d = A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}$ back into

[10] Similar modeling approach was taken by Strulik et al (2013) and Prettner (2014).

(1a) yields the following expression for total output

$$Y_t = A^{\frac{1}{1-\alpha}} N_t L \left(\frac{\alpha^2}{\lambda(1+r)} \right)^{\frac{\alpha}{1-\alpha}} \tag{16}$$

The saving and investment equations modify to

$$S_t = \frac{(1-\alpha) A^{\frac{1}{1-\alpha}} N_t L \left(\frac{\alpha^2}{\lambda(1+r)} \right)^{\frac{\alpha}{1-\alpha}}}{1+\rho^{-1}} \tag{17}$$

$$I_t = N_{t+1} \left[\eta + A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda(1+r)} \right)^{\frac{1}{1-\alpha}} \right] \tag{18}$$

Equalizing aggregate saving and aggregate investment yields the stationary growth rate

$$1+g = \frac{1-\alpha}{1+\rho^{-1}} \frac{A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda(1+r)} \right)^{\frac{\alpha}{1-\alpha}}}{\eta + A^{\frac{1}{1-\alpha}} L \left(\frac{\alpha^2}{\lambda(1+r)} \right)^{\frac{1}{1-\alpha}}} \tag{19}$$

We define $\psi \equiv \frac{\alpha^2}{\lambda(1+r)}$ and plugging it into (19) to re-write the growth equation

$$1+g = \frac{1-\alpha}{1+\rho^{-1}} \frac{\psi^{\frac{\alpha}{1-\alpha}}}{\frac{\eta}{A^{\frac{1}{1-\alpha}} L} + \psi^{\frac{1}{1-\alpha}}} \tag{20}$$

Proposition 4 *For production with differentiated physical capital, the growth maximizing patent breadth is always incomplete*

Proof. Differentiating (20) for ψ reveals that the growth rate is increasing with ψ, if $\frac{\alpha}{1-\alpha} \frac{\eta}{A^{\frac{1}{1-\alpha}} L} > \psi^{\frac{1}{1-\alpha}}$, that is $\frac{\alpha}{1-\alpha} \frac{\eta}{A^{\frac{1}{1-\alpha}} L} > \left(\frac{\alpha^2}{\lambda(1+r^*)} \right)^{\frac{1}{1-\alpha}}$. Hence, under this condition the growth rate is decreasing in λ. We evaluate the latter condition, for $\lambda = 1$, by plugging into it the interest rate for one-period patent duration, given in (15a). For this interest rate the latter inequality, does not hold, but rather it turns to equality. Hence, because the interest rate under infinite patent duration that is given in (15) is higher, the latter inequality always holds. ■

4.2 Patent length

We focus here on the effect of patent duration on growth given that patent breadth protection is complete, $\lambda = 1$. The discussion in subsection 3.2.1 suggests that Proposition 4 implies also that growth is always maximized with less than infinite patent duration: the effect of marginal shortening of patent duration on ψ, is equivalent to the effect of marginal decrease in λ[11]. In addition shortening patent duration decreases the average fix-cost per variety, denoted $\overline{v}(T)$ in the discussion above, which by itself works to enhance growth. Nevertheless, a formal proof of this

[11] As both decrease the average price of machines and the interest rate.

conjecture turns out to be intractable. Hence, we will focus here on comparing growth under one-period patent duration and under infinite patent duration, for $\lambda = 1$. Under one period patent length, the relevant interest rate is the one defined in equation (15a), and the growth equation modifies to the following implicit equation

$$1+g=\frac{1-\alpha}{1+\rho^{-1}}\frac{g\left(\frac{\alpha^2}{1+r}\right)^{\frac{\alpha}{1-\alpha}}+\left(\frac{\alpha}{1+r}\right)^{\frac{\alpha}{1-\alpha}}}{g\left(\frac{\eta}{A^{\frac{1}{1-\alpha}}L}+\left(\frac{\alpha^2}{1+r}\right)^{\frac{1}{1-\alpha}}\right)+\left(\frac{\alpha}{1+r}\right)^{\frac{1}{1-\alpha}}} \tag{21}$$

Elaborating (21) yields the xplicit expression for the growth rate under one period duration

$$1+g=\frac{1-\alpha}{1+\rho^{-1}}A\left(\frac{L}{\eta}\right)^{1-\alpha}\alpha\left(\frac{1}{\alpha}-1\right)^{1-\alpha}=\frac{1-\alpha}{1+\rho^{-1}}\frac{1+r}{\alpha} \tag{21a}$$

Proposition 5 *For production with differentiated investment goods, one period patent duration yields lower growth rate than infinite duration if* $\frac{\eta}{A^{\frac{1}{1-\alpha}}L}$ *is sufficiently low.*

Proof. The proof derives from comparing the growth rate under one period patent duration given in (21a) with the growth rate under infinite patent duration given in (20), for complete patent breadth, i.e. $\lambda = 1$. For sufficiently low $\frac{\eta}{A^{\frac{1}{1-\alpha}}L}$, the growth rate in (20) approaches $\frac{1-\alpha}{1+\rho^{-1}}\frac{1+r}{\alpha^2}$, which is greater than the one in (21a); recall that the interest rate in (20) is higher. ■

It can be also shown that under one period patent duration growth is maximized with full patent breadth protection.

4.3 Welfare

Our welfare analysis focuses on the effect of patent breadth protection in growth under production with physical capital and infinite patent duration. Welfare is defined as the present value of the life time utility for the living generation and all future generations, under the private discount rate:

$$W\equiv\sum_{t=0}^{\infty}\rho^t U_t \tag{22}$$

Where U_t is the life utility of the representative consumer who was born in period t, given in equation (3). To maintain tractability we confine to patent breadth protection. Plugging in the explicit expressions for c_1 and c_2 in the lifetime utility function (3) yields the following indirect utility function

$$U_t=\ln\left[\frac{(1-\alpha)N_tA^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{\lambda(1+r)}\right)^{\frac{\alpha}{1-\alpha}}}{1+\rho}\right]+\rho\ln\left[\frac{\rho(1-\alpha)N_tA^{\frac{1}{1-\alpha}}\left(\frac{\alpha^2}{\lambda(1+r)}\right)^{\frac{\alpha}{1-\alpha}}}{1+\rho}(1+r)\right] \tag{23}$$

Plugging (16) into (15), and elaborating the resulting geometric sequence, yields the following expression for the present value of the life-time utility for the current and all future generations, given the initial variety span N_0:

$$W = \frac{U_0}{1-\rho} + \frac{\rho\,(1+\rho)\ln(1+g)}{1-\rho} \tag{24}$$

Proposition 6 *Incomplete patent breadth protection is always welfare improving*

Proof. By Proposition 4, growth rate g is always maximized under incomplete breadth protection. Hence, by (25), a sufficient condition for incomplete breadth protection to maximize welfare is having $\frac{\partial U_0}{\partial \lambda} < 0$ when evaluated at $\lambda = 1$. The sign of the derivative of (24), given N_0, is given by $-\frac{(1+\rho)\alpha}{\lambda} + \frac{\frac{\partial(1+r)}{\partial\lambda}}{1+r}\left[\rho(1-2\alpha)-\alpha\right]$. Lemma 1 states that the expression increasing $\frac{\partial(1+r)}{\partial\lambda}$ is zero when evaluated at $\lambda = 0$. Hence, the sign of $\frac{\partial U_0}{\partial \lambda} < 0$ when evaluated at $\lambda = 1$, that is marginal loosening of patent breadth is welfare improving. ■

4.4 CRRA Utility

Finally, we turn now to consider the implication of the general CRRA instantaneous utility to our previous result, considering the following lifetime utility form:

$$U = \frac{c_t^{1-\theta}}{1-\theta} + \rho\frac{c_{t+1}^{1-\theta}}{1-\theta} \tag{25}$$

where $\frac{1}{\theta}$ is the elasticity of inter-temporal substitution, and for $\theta = 1$ (26) falls back to the logarithmic form (3). The modified solution for the standard optimal saving problem is $s_t = \frac{w_t}{1+\rho^{-\frac{1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}$. Hence, aggregate saving now is $S_t = \frac{w_t L}{1+\rho^{-\frac{1}{\theta}}(1+r)^{\frac{\theta-1}{\theta}}}$ .Substituting the explicit expressions for w_t into S_t and equalizing to aggregate investment given in (18) yields the growth equation

$$1+g = \frac{1-\alpha}{1+\frac{1}{\rho}\left(1+r\right)^{1-\frac{1}{\theta}}}\frac{A^{\frac{1}{1-\alpha}}L\left(\frac{\alpha^2}{\lambda(1+r)}\right)^{\frac{\alpha}{1-\alpha}}}{\eta + A^{\frac{1}{1-\alpha}}L\left(\frac{\alpha^2}{\lambda(1+r)}\right)^{\frac{1}{1-\alpha}}} \tag{26}$$

As it is well known, in the standard OLG framework the effect of interest rate on saving depends on the inter-temporal elasticity of substitution: it is positive (negative) if $\theta < 1$ ($\theta > 1$). Hence, because the interest rate is increasing with patent protection, the positive impact of decreasing patent breadth on growth is diminishing with the inter-temporal elasticity of substitution. More specifically, for $\theta < 1$ all our results remain (and will hold for a larger set of parameters) as a decrease in the interest rate by itself stimulates saving and investment (this is an additional effect was not induced under the logarithmic utility form). However, as θ increases beyond one, the decrease in the interest rate due to loosening patent breadth protection will work to hinder growth, countering the positive effects that were defined in Propositions 1-3. For sufficiently high value of θ this direct interest effect may entirely dominate the over all impact of loosening patent protection on innovation and growth. Nevertheless, the empirical literature commonly suggests that θ is lower than one, and thus supporting the relevance of our main findings[12]. The welfare analysis for $\theta \neq 1$ turns out being intractable.

5 Conclusions

This work contributes to the literature on patent policy and economic growth by analyzing the growth and welfare implication of patent length and breadth in the overlapping generations framework of finitely living agents. The OLG demographic structure gives room to inter-generational trade in patents and life-cycle saving. These elements, which are proved to be relevant to patent policy are abstracted in large by the current literature on patents and growth, which is written for infinitely living agents.

Our comparative analysis implies that these two elements provide a case for incomplete patent protection that is both growth and welfare enhancing, in contrast with the results obtained in the analyzes of infinitely living agents. Compared with the few previous studies on patent policy written in the OLG framework, our contribution is twofold: (a) highlighting the dependency of patent policy implications on the assumed nature of the differentiated goods, whether they are consumption or intermediate goods or investment goods (b) exploring the uneven effect of patent breadth and duration on growth.

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